# The problem of time in quantum cosmology and the definition of a time operator

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# Outline

# Cosmology as a geodesic problem

# The mini-superspace system

Initial action

$$S = \int_{\Omega} d^4x\sqrt{-g}R + S_m \tag{1}$$

Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \tag{2}$$

Impose a certain group of isometries

$$ds^2 = -N^2(t)dt^2 + \gamma_{\mu\nu}(t)\sigma_i^{\mu}(x)\sigma_j^{\nu}(x)dx^i dx^j \tag{3}$$

Equations (2) reduce to ODEs.
Under (3)

$$S = \mathcal{V}_0 S_{red}[t] = \int_a^b L(t)dt \tag{4}$$

Prototype mini-superspace Lagrangian

$$L = \mathcal{V}_0 \left( \frac{1}{2N(t)} G_{\alpha\beta}(q(t)) \dot{q}^\alpha \dot{q}^\beta - N(t) V(q(t)) \right) \tag{5}$$

$\alpha, \beta = 1, ..., d = \dim G_{\alpha\beta}$

Hamiltonian description

$$H_T = N\mathcal{H} + u_N p_N \tag{6}$$

where

$$p_N \approx 0 \tag{7a}$$

$$\mathcal{H} = \frac{1}{2\mathcal{V}_0} G^{\alpha\beta} p_\alpha p_\beta + \mathcal{V}_0 V(q) \approx 0 \tag{7b}$$

Dirac's prescription for quantization

$$\hat{p}_N \Psi = 0, \qquad \hat{\mathcal{H}} \Psi = 0 \tag{8}$$

# The equivalent geodesic problem

reparametrize the lapse $N \to n = NV(q)/\mathcal{V}_0$

$$L = \frac{1}{2n} V(q) G_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu - n \mathcal{V}_0^2 \tag{9}$$

Geodesic Lagrangian in the einbein formalism

$$L = \frac{1}{2e} g_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu - e \frac{m^2 c^2}{2} \tag{10}$$

metric of the equivalent geodesic system

$$g_{\mu\nu}(q) = V(q) G_{\mu\nu}(q) \tag{11}$$

# Point symmetries of $L$

$$L = \frac{1}{2e} g_{\mu\nu}(q)\dot{q}^\mu \dot{q}^\nu - e\frac{m^2 c^2}{2}$$

Known symmetries $X = \xi^\mu(q)\frac{\partial}{\partial q^\mu}$:

| $m = 0$ | $m \neq 0$ |
|---|---|
| CKVs<br>$\mathcal{L}_\xi g_{\mu\nu} = 2\omega(q) g_{\mu\nu}$ | KVs<br>$\mathcal{L}_\xi g_{\mu\nu} = 0$ |

Conserved charges: $I = \xi^\mu \frac{\partial L}{\partial \dot{q}^\mu} = \xi^\mu p_\mu$

Explicit symmetry breaking when $m \neq 0$

$$\text{KVs} \subset \text{CKVs}$$

What happens with the broken symmetries when $m \neq 0$?

# Hamiltonian in the einbein formalism

Total Hamiltonian from the Dirac-Bergmann algorithm

$$H_T = \frac{e}{2}\mathcal{H} + u_e p_e, \tag{12}$$

with the first class constraints

$$\mathcal{H} = g^{\mu\nu} p_\mu p_\nu + m^2 c^2 \approx 0 \quad \text{and} \quad p_e \approx 0 \tag{13}$$

"$\approx$" signifies a weak equality.

A function $\mathcal{H} = \mathcal{H}(q, p)$ is weakly equal to zero, i.e. $\mathcal{H} \approx 0$, if:

- $\mathcal{H}(q, p) = 0$ but,
- $\left(\frac{\partial \mathcal{H}}{\partial q}, \frac{\partial \mathcal{H}}{\partial p}\right) \neq 0$

# Kuchař’s Conditional Symmetries

[K. V. Kuchař, J. Math. Phys. **23** (1982), 1647-1661]

Conserved charge $I = I(q, p)$ in a constrained system.

In place of

$$\frac{dI}{d\tau} = 0 \Rightarrow \{I, H_T\} = 0 \tag{14}$$

demand

$$\frac{dI}{d\tau} \approx 0 \Rightarrow \{I, H_T\} = \sigma\mathcal{H}. \tag{15}$$

The solution of (14) is contained in (15).

No new integrals of motion for the geodesic system for $I = I(q, p)$.

This changes if you consider $I = I(\tau, q, p)$.

# Nonlocal conserved charges

# Introducing Nonlocal Conserved Charges

[N.D., P.A. Terzis and T. Christodoulakis, Phys. Rev. D **99** (2019), 104061],

[N.D., Phys. Rev. D **106** (2022), 024043]

Let us take

$$I(\tau, q, p) = \xi^\mu p_\mu + m^2 c^2 \int e(\tau)\omega(q(\tau))d\tau, \tag{16}$$

with

$$\mathcal{L}_\xi g_{\mu\nu} = 2\omega(q) g_{\mu\nu}. \tag{17}$$

Then,

$$\frac{dI}{d\tau} = \frac{\partial I}{\partial \tau} + \{\xi^\mu p_\mu, H_T\} = e\,\omega(q)\mathcal{H} \approx 0 \tag{18}$$

For every proper CKV $\xi$ we have a nonlocal integral of motion $I$.

$$I = \xi^\mu p_\mu + m^2 c^2 \int e(\tau)\omega(q(\tau))d\tau \quad , \quad \mathcal{L}_\xi g_{\mu\nu} = 2\omega(q) g_{\mu\nu} \qquad (19)$$

Connection to known results:

- If $\xi$ is a Killing vector, $\omega = 0$

$$I = \xi^\mu p_\mu \quad , \quad \mathcal{L}_\xi g_{\mu\nu} = 0$$

- If the particle is massless, $m = 0$

$$I = \xi^\mu p_\mu \quad , \quad \mathcal{L}_\xi g_{\mu\nu} = 2\omega(q) g_{\mu\nu}$$

- If we have an affine parametrization e.g. $e = 1/m$ and $\xi$ is a homothetic vector, i.e. $\omega = 1$

$$I = \xi^\mu p_\mu + m\, c^2 \tau$$

# Time as a phase space function

In the proper time gauge $e = 1/m$,

$$I = \xi^\mu p_\mu + m\,c^2 \tau. \tag{20}$$

If $Q = \xi^\mu p_\mu$, then, up to an additive constant,

$$\tau = -\frac{1}{m\,c^2} Q = -\frac{1}{m\,c^2} \xi^\mu p_\mu. \tag{21}$$

# Quantum time operator

# Quantization

Classical constraints: $p_n \approx 0$ and

$$\mathcal{H} = g^{\mu\nu} p_\mu p_\nu + m^2 c^2 \approx 0 \tag{22}$$

Quantum operator for the kinetic part (conformal Laplacian)

$$\widehat{L}_g = \frac{1}{\mu}\partial_\alpha \left(\mu\, g^{\alpha\beta}\partial_\beta\right) - \frac{d-2}{4(d-1)} R, \tag{23}$$

measure $\mu = \sqrt{-g}$.
Dirac's quantization conditions: $\widehat{p}_n \Psi = 0$ and $\widehat{\mathcal{H}}\Psi = 0$.
General linear, first order Hermitian operator

$$\begin{aligned} \widehat{Q} &= -\frac{\mathrm{i}\hbar}{2\mu}\left(\mu\, \xi^\alpha \partial_\alpha + \partial_\alpha\left(\mu\, \xi^\alpha\right)\right) \\ &= -\mathrm{i}\hbar \xi^\alpha \nabla_\alpha - \frac{\mathrm{i}\hbar}{2}\nabla_\alpha \xi^\alpha . \end{aligned} \tag{24}$$

# Symmetries of the conformal Laplacian

For any CKV $\xi$

$$\widehat{D}_\xi = \xi^\alpha \nabla_\alpha + \frac{d-2}{2d} \left( \nabla_\alpha \xi^\alpha \right) \tag{25a}$$

$$\widehat{\delta}_\xi = \xi^\alpha \nabla_\alpha + \frac{d+2}{2d} \left( \nabla_\alpha \xi^\alpha \right), \tag{25b}$$

satisfy

$$\widehat{L}_g \widehat{D}_\xi = \widehat{\delta}_\xi \widehat{L}_g. \tag{26}$$

With the above relation it can be seen that

$$\widehat{\mathcal{H}} \widehat{Q} \Psi = 2 \mathrm{i} \hbar \, m^2 c^2 \Psi \tag{27}$$

or, for a wave function invariant under the constraint $\widehat{\mathcal{H}} \Psi = 0$

$$[\widehat{\mathcal{H}}, \widehat{Q}] \Psi = 2 \, \mathrm{i} \, \hbar \, m^2 c^2 \Psi. \tag{28}$$

# Quantum time operator

Defining from the classical equivalent $\tau = -\frac{1}{m\,c^2} Q$,

$$\widehat{T} := -\frac{1}{m\,c^2}\widehat{Q}. \tag{29}$$

it follows that

$$[\widehat{T}, \widehat{\mathcal{H}}/m]\Psi = 2\mathrm{i}\hbar\Psi. \tag{30}$$

$\mathcal{H}/m \rightarrow$ units of energy

$$\frac{1}{m}\widehat{\mathcal{H}}\Psi = 0 \Rightarrow \widehat{E}\Psi = m\,c^2\Psi, \tag{31}$$

where $\widehat{E} = \hbar^2 \widehat{L}_g$.
Time-energy uncertainty relation?

$$[\widehat{E}, \widehat{T}]\Psi = 2\mathrm{i}\hbar\,\Psi \Rightarrow (\Delta E)\,(\Delta T) \geq \hbar. \tag{32}$$

# Examples

# Outline

# Relativistic Free particle

Classical Hamiltonian constraint:

$$\mathcal{H} = -p_0^2 + \vec{p}^{\,2} + m^2c^2 \approx 0 \tag{33}$$

Klein-Gordon equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\Psi = 0. \tag{34}$$

reduces to Schrödinger equation under the ansatz

$$\Psi = e^{-\mathrm{i}mc^2t/\hbar}\psi(t,x,y,z) \tag{35}$$

and the assumption $c^2 >> 1$

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi. \tag{36}$$

# Relativistic Free particle

Homothety of Minkowski metric

$$\xi = t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}. \tag{37}$$

Time operator

$$\widehat{T} = \mathrm{i}\frac{\hbar}{m\,c^2}\left(t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i} + \frac{1}{2}\right). \tag{38}$$

Under the same assumptions as before

$$\widehat{T}\Psi \simeq t\Psi. \tag{39}$$

# Outline

# FLRW spacetime with a massless scalar field

Spacetime metric

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \tag{40}$$

Minisuperspace Lagrangian

$$L = \mathcal{V}_0 \left( -\frac{3a\dot{a}^2}{\kappa N} + \frac{a^3 \dot{\phi}^2}{N} + N \frac{3ka}{\kappa} \right) \tag{41}$$

Perform a reparametrization $N = \kappa \mathcal{V}_0 n/(6a)$

$$L = \frac{6}{n} \left( -\frac{3a^2 \dot{a}^2}{\kappa^2} + \frac{a^4 \dot{\phi}^2}{\kappa} \right) + \frac{n}{2} k \mathcal{V}_0^2. \tag{42}$$

# Classical setting

Mini-superspace metric

$$G_{\mu\nu} = \begin{pmatrix} -\frac{36a^2}{\kappa^2} & 0 \\ 0 & \frac{12a^4}{\kappa} \end{pmatrix} \tag{43}$$

has the homothecy $\xi = \frac{a}{2}\frac{\partial}{\partial a}$.

Hamiltonian constraint:

$$\mathcal{H} = \frac{\kappa p_\phi^2}{24a^4} - \frac{\kappa^2 p_a^2}{72a^2} - \frac{k\mathcal{V}_0^2}{2} \approx 0 \tag{44}$$

proper time variable

$$T = \frac{1}{k\mathcal{V}_0^2}\frac{a}{2}p_a \tag{45}$$

# Quantization

WDW equation $\hat{\mathcal{H}}\Psi = 0$:

$$\hat{\mathcal{H}}\Psi = \frac{\kappa^2\hbar^2}{72a^2}\frac{\partial^2\Psi}{\partial a^2} + \frac{\kappa^2\hbar^2}{72a^3}\frac{\partial\Psi}{\partial a} - \frac{\kappa\hbar^2}{24a^4}\frac{\partial^2\Psi}{\partial\phi^2} - \frac{1}{2}k\mathcal{V}_0^2\Psi = 0 \qquad (46)$$

Time operator:

$$\widehat{T} = -\frac{\mathrm{i}\hbar}{k\mathcal{V}_0^2}\left(\frac{a}{2}\frac{\partial}{\partial a} + 1\right) \qquad (47)$$

$$[\hat{T}, \hat{\mathcal{H}}]\Psi = \mathrm{i}\hbar\Psi, \quad \forall\ \Psi \text{ satisfying } \hat{\mathcal{H}}\Psi = 0 \qquad (48)$$

Make a canonical transformation

$$\frac{1}{2k\mathcal{V}_0^2} a\, p_a = T, \quad -2k\mathcal{V}_0^2 \ln a = p_T. \tag{49}$$

Hamiltonian constraint becomes

$$\frac{\kappa}{24} p_\phi^2 - \frac{\kappa^2 k^2 \mathcal{V}_0^4}{18} T^2 - \frac{k\mathcal{V}_0^2}{2} e^{-\frac{2p_T}{k\mathcal{V}_0^2}} \approx 0 \tag{50}$$

Consider: $a \sim 1 \Rightarrow p_T << 1, \qquad \dot{a}^2 \sim 0 \Rightarrow T^2 \sim 0$

Approximate Hamiltonian constraint

$$\frac{\kappa}{24} p_\phi^2 - \frac{k\mathcal{V}_0^2}{2} + p_T \approx 0 \tag{51}$$

# Conclusion

# Concluding remarks

- Use of a non-local symmetry, which is related to the proper time
- Construct a Hermitian operator based on this symmetry
- This new time operator satisfies a canonical commutation relation with $\widehat{\mathcal{H}}$, given that $\Psi$ is a physical state $\widehat{\mathcal{H}}\Psi = 0$.
- Possible time-energy uncertainty relation, where $E$ is the rest energy.
- Investigation of future applications in cosmology.

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