Superconducting multi-vortices and a novel BPS bound in chiral perturbation theory

CosmoConce y Partículas, 2025
Concepción, Chile

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Based on:
*F. Canfora, M.L. and A. Vera, JHEP **10**, 224 (2024).*

Advantages of the availability of BPS bounds

The appearance of a BPS configuration arises from a delicate balance between opposite forces (typically, when attractive and repulsive interactions of the theory of interest are of the same order), one of the most important examples being BPS vortices in superconductors.

Despite the fact that these configurations appear at special points in parameters space, they have fundamental importance for several reasons. Just a few of them are:

- Signal of a transition from one type of behavior to a different type of behavior.
- The physical effects generated by BPS configurations are non-perturbative in nature.
- There is a very powerful tool to describe the low-energy dynamics of these BPS configurations in terms of geodesics on moduli space.
- BPS configurations are topologically stable: these solutions cannot be destroyed by quantum/thermal fluctuations.

Motivation

What we know:

- From the Ginzburg-Landau theory:

  There is a phase transition between Type-I and Type-II superconductors at a critical value of the phase of the Higgs field. At the transition point a BPS bound is saturated by multi-vortex configurations, and where the magnetic flux density plays the role of the topological charge density. *A. A. Abrikosov, Sov. Phys. JETP **5**, 1174-1182 (1957)*, *H. B. Nielsen and P. Olesen, Nucl. Phys. B 61, 45-61 (1973)*.

- From QCD at finite isospin chemical potential:

  At finite μ_I phase transitions occur. At low isospin density, the ground state is a pion condensate, while at higher density a Fermi liquid with Cooper pairing should appear. *D.T. Son, M. A. Stephanov, Phys. Rev. Lett. 86 (2001) 592-595*.

The idea:

- Since μ_I is responsible for the Cooper pairing, playing a similar role to the Higgs coupling in GL, one would expect that at a special value of μ_I it should be possible to saturate a BPS bound providing some suitable topological charge density.

- QCD at low energies: It can be described by Chiral Perturbation Theory through an action obtained from the momentum expansion. This theory allows topological soliton solutions. *S. Scherer, Adv. Nucl. Phys., vol. 27, p. 277, 2003*.

- From ChPT coupled to the Maxwell theory, one could derive a BPS bound for a critical value of the isospin chemical potential in such a way that multi-vortices solutions should appear.

Gauged - Chiral Perturbation Theory (ChPT)

Up to order $\mathcal{O}(p^2)$, is described by the action

$$I[U,A] = \frac{1}{4e^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(K\,\mathrm{Tr}\left[R^{\mu}R_{\mu}\right] - F_{\mu\nu}F^{\mu\nu}\right) \ , \tag{1}$$

$$R_{\mu} = U^{-1}D_{\mu}U = R^{j}_{\mu}t_j \ , \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \ , \quad t_j = i\sigma_j \ .$$

Here $U(x) \in SU(2)$ is the pionic field, A_{μ} is the Maxwell potential. The couplings constants are $K = (\frac{ef_{\pi}}{2})^2$, being f_{π} the pions decay constant and e the electric charge; $f_{\pi} = 93\,\mathrm{MeV}$ and $e = \sqrt{4\pi\alpha} = 0,303$.
In Eq. (1), D_{μ} denotes the covariant derivative, defined as

$$D_{\mu}U = \nabla_{\mu}U + A_{\mu}\left[t_3, U\right] \ .$$

The isospin chemical potential can be introduced to the model through the covariant derivative in the following form

$$D_{\mu}U \rightarrow D_{\mu}U + \mu_I[t_3, U]g_{\mu t} \ ,$$

where μ_I is the value of isospin chemical potential.

The matter fields

The pionic field in the exponential representation is written as

$$U = \cos(\alpha)\mathbf{1} + \sin(\alpha) n_i t^i \ , \tag{2}$$
$$n_i = \{\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta\} \ ,$$

where $\alpha = \alpha(x^\mu)$, $\Theta = \Theta(x^\mu)$, $\Phi = \Phi(x^\mu)$ are the three degrees of freedom of the U field. In terms of this parametrization, the covariant derivative reads

$$D_\mu \alpha = \partial_\mu \alpha \ , \quad D_\mu \Theta = \partial_\mu \Theta \ , \quad D_\mu \Phi = \partial_\mu \Phi - 2A_\mu \ .$$

One can see that the scalar degree of freedom $\Phi(x^\mu)$ plays the role of the phase of the complex Higgs field in the GL theory.
For multi-vortices with quantized magnetic field along the third spatial direction, the natural Ansatz is

$$\alpha = \alpha(x_1, x_2) \ , \quad \Phi = \Phi(x_1, x_2) \ , \quad \Theta = \frac{\pi}{2} \ , \tag{3}$$
$$A_\mu dx^\mu = A_1 \, dx_1 + A_2 \, dx_2 \ , \qquad A_i = A_i(x_1, x_2) \ . \tag{4}$$

The Gibbs free energy

With the above Ansatz, the free energy density F of the system becomes

$$e^2 F = \frac{K}{2}\left\{\left(\vec{\nabla}\alpha\right)^2 + \sin^2(\alpha)\left(\vec{D}\Phi\right)^2\right\} + \frac{1}{2}(\vec{B}^2) - 2K\mu_I^2 \sin^2(\alpha) \ . \tag{5}$$

In order to derive a BPS bound it is convenient to rewrite Eq. (5) as follows:

$$e^2 F = \frac{K}{2}\left\{\left(\vec{\nabla}\alpha\right)^2 + \sin^2(\alpha)\left(\vec{D}\Phi\right)^2\right\} + \frac{1}{2}(\vec{B}^2) + 2K\mu_I^2 \cos^2(\alpha) - 2K\mu_I^2 \ .$$

The above expression would be positive definite if one could eliminate the last constant term, so that, the appropriate thermodynamical potential in this case is not the free energy $\mathcal{F}$,

$$\mathcal{F} = \int d^3x \, F \ ,$$

but rather the Gibbs free energy $\mathcal{G}$,

$$\mathcal{G} = \mathcal{F} + PV \ , \quad \mathcal{G} = \int d^3x \, G \ ,$$

being V the volume and P the pressure fixed by the chemical potential, namely $P = 2K\mu_I^2$. Thus, the Gibbs free energy density G reads

$$e^2 G = \frac{K}{2}\left\{\left(\vec{\nabla}\alpha\right)^2 + \sin^2(\alpha)\left(\vec{D}\Phi\right)^2\right\} + \frac{1}{2}(\vec{B}^2) + 2K\mu_I^2 \cos^2(\alpha) \ . \tag{6}$$

BPS completion

Conveniently, after gauging away the phase Φ, we can rewrite the Gibbs free energy density as

$$\begin{aligned} e^2 G = & \frac{K}{2}\left((\partial_1 \alpha + \sin(\alpha) A_2)^2 + (\partial_2 \alpha - \sin(\alpha) A_1)^2\right) + \frac{1}{2}\left(B_z - 2\mu_I \sqrt{K}\cos(\alpha)\right)^2 \\ & - K \sin\alpha (\partial_1 \alpha A_2 - \partial_2 \alpha A_1) + 2\mu_I \sqrt{K}\cos(\alpha) B_z \ . \end{aligned}$$

The last two terms conform a total derivative for a critical value of the isospin chemical potential

$$\mu_I^{\mathrm{c}} = \sqrt{K} = \frac{e f_\pi}{2} = 14,1\,\mathrm{MeV} \ . \tag{7}$$

In fact, for this value

$$-K \int dx^1 dx^2 \left[\sin(\alpha)(\partial_1 \alpha\, A_2 - \partial_2 \alpha\, A_1) - \cos(\alpha) B_z\right] = \int d\omega \ , \qquad \omega = \cos(\alpha)(A_1 dx^1 + A_2 dx^2) \ .$$

This is the magnetic flux density but dressed by the hadronic profile!

BPS completion

Thus, the free energy density is minimized when the following BPS first order equations are satisfied

$$\partial_1 \alpha + \sin(\alpha)\, A_2 = 0 \ , \tag{8}$$

$$\partial_2 \alpha - \sin(\alpha)\, A_1 = 0 \ , \tag{9}$$

$$B_z - 2K \cos(\alpha) = 0 \ , \tag{10}$$

and the following BPS bound emerges

$$\mathcal{G} \geq \int_{\partial\Sigma} d\omega \ ,$$

where $\partial\Sigma$ is usually taken as the S^1 circle at spatial infinity. Note that, for small α, the BPS

system reduces exactly to the BPS system of multi-vortices in critical superconductors. In fact, $|\alpha| \ll 1$, we obtain the well-known system

$$\partial_1 \alpha + \alpha\, A_2 = 0 \ ,$$

$$\partial_2 \alpha - \alpha\, A_1 = 0 \ ,$$

$$B_z - K(1 - \frac{\alpha^2}{2}) = 0 \ .$$

A single vortex

Let us consider the Ansatz for a single vortex

$$\alpha = \alpha(r) \ , \quad A_r = A_z = 0 \ , \quad A_\theta = A(r) \ , \quad \Phi = n\theta \ ,$$
$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \ ,$$

together with the boundary conditions; $\alpha \underset{r\to\infty}{\to} \frac{\pi}{2}$, $\alpha \underset{r\to 0}{\to} \pi$, $\Phi(r,\theta) = \Phi(r,\theta+2\pi) + 2n\pi$, $n \in \mathbb{N}$.
The Gibbs free energy of the system is

$$\begin{aligned} \frac{\mathcal{G}}{L} =& 2\pi \int \frac{1}{2e^2 r} \{K(r\alpha' + \sin(\alpha)(n-2A))^2 + (A' + \bar{\mu} r \cos(\alpha))^2\} dr \\ &- \frac{2\pi}{e^2} \int \Big\{ K \sin(\alpha)(n-2A)\alpha' + \bar{\mu}\cos(\alpha) A' \Big\} \, dr \ . \end{aligned}$$

The second term is a total derivative for $\mu_I^c = \sqrt{K}$;

$$-K\sin(\alpha)(n-2A)\alpha' - \bar{\mu}\cos(\alpha)A' = \{K\cos(\alpha)(n-2A)\}' \ .$$

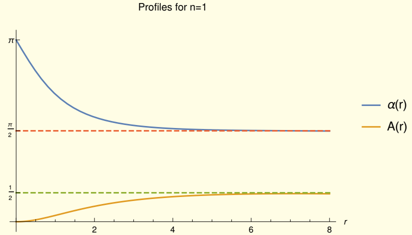
It follows that the Gibbs free energy is minimized when the following BPS equations are satisfied

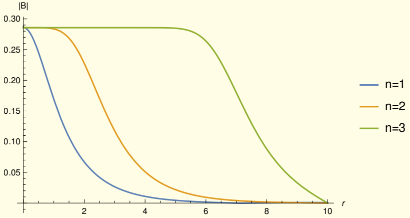
$$r\alpha' + \sin(\alpha)(n-2A) = 0 \ , \tag{11}$$
$$A' + 2Kr\cos(\alpha) = 0 \ . \tag{12}$$

The topological charge of this configuration reads

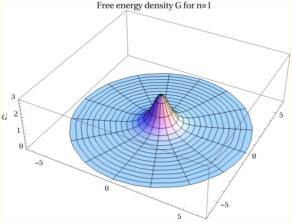
$$Q = \frac{2\pi}{e^2} \int \{K\cos(\alpha)(n-2A)\}' dr = \frac{2\pi K}{e^2} n \ .$$

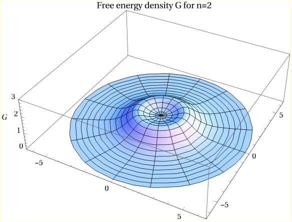
A single vortex

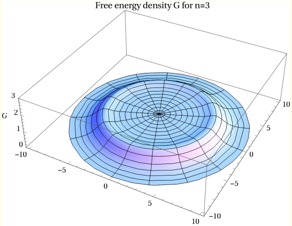




A single vortex







Multi-vortices at critical values

The novel BPS bound has many relevant properties.

1. The topological charge density is not simply the magnetic flux density, it is a "dressed magnetic flux" modulated by the hadronic profile. *F. Canfora, JHEP 11 (2023) 007.*

2. The fact that these multi-solitons are minima of the Gibbs free energy (and not of the free energy) means that they can be realized at $P = 2K(\mu_I^c)^2 = 858.474\,(\mathrm{MeV})^4$.

3. The present BPS bound allows to find very easily the maximum value for the magnetic field beyond which the condensate ceases to exist. Indeed, looking at Eq. (10), one gets

$$B_{\max} = 2K = 2\left(\frac{ef_\pi}{2}\right)^2 = 397,03\,(\mathrm{MeV})^2 = 2,04 \times 10^{14}\ \mathrm{G}\ . \tag{13}$$

Note that this maximum value is of the order of what is expected for magnetars ($\sim 10^{13}$ G to $\sim 10^{15}$ G).

4. The magnetic field is generated by a self-sustained current, given by

$$J_\mu = 2K \sin^2(\alpha) D_\mu \Phi\ . \tag{14}$$

This current is not-null even when the electromagnetic field is suppressed. In fact, there is a persistent current generated by the coupling with pions, given by $J_\mu^{(0)} = 2K \sin^2(\alpha)\partial_\mu \Phi$.

Multi-vortices at critical values

5. As expected, the first order BPS equations imply the following second order system

$$\triangle\alpha - \sin(\alpha)\cos(\alpha)\left((\vec{D}\Phi)^2 - 4\mu_I^2\right) = 0 \ ,$$

$$\triangle\Phi - 2\ \vec{\nabla}\cdot\vec{A} + 2\cot(\alpha)\vec{\nabla}\alpha\cdot\vec{D}\Phi = 0 \ ,$$

$$\partial_j F^{ij} - 2K\sin^2(\alpha)D^i\Phi = 0 \ ,$$

obtained through the variation of the free energy density with respect to the field α, Φ and A_μ.

6. The present formalism can be applied even when the gauged ChPT includes a pions mass term S. B. Gudnason and M. Nitta, Phys. Rev. D **94**, no.6, 065018 (2016).

In particular, considering

$$\mathcal{L}_{\text{mass}} = -\frac{2Km_\pi^2}{e^2}\left(1 - \frac{1}{4}\text{Tr}[U]^2\right) = \frac{2Km_\pi^2}{e^2}\sin^2(\alpha) \ ,$$

the free energy density of the system becomes

$$e^2\mathcal{F} = \frac{K}{2}\left\{\left(\vec{\nabla}\alpha\right)^2 + \sin^2(\alpha)\left(\vec{D}\Phi\right)^2\right\} + \frac{1}{2}(\vec{B}^2) + 2K\left(\mu_I^2 - m_\pi^2\right)\cos^2(\alpha) - 2K\left(\mu_I^2 - m_\pi^2\right) \ .$$

Consequently, the inclusion of the mass term for the pions manifests itself in a shift:

$$\mu_I^2 \to \mu_I^2 - m_\pi^2 \ .$$

Hence, all the previous results still hold. The critical value for the isospin chemical potential when the pions mass is taken into account reads

$$\mu_I^{\text{c}} = \sqrt{K + m_\pi^2} < 1.1 m_\pi \ .$$

Thank you!