Mixing “Magnetic” and “Electric” Ehlers-Harrison transformations: the electromagnetic swirling spacetime and novel type I backgrounds

Keanu Müller

In colaboration with: José Barrientos, Adolfo Cisterna, Ivan Kolár, Marcelo Oyarzo and Konstantinos Pallikaris.

Eur.Phys.J.C **84 (2024) 7, 724,**
arXiv:2401.02924v2 [gr-qc]

UNIVERSIDAD DEL BÍO-BÍO

November, 2025

Outline

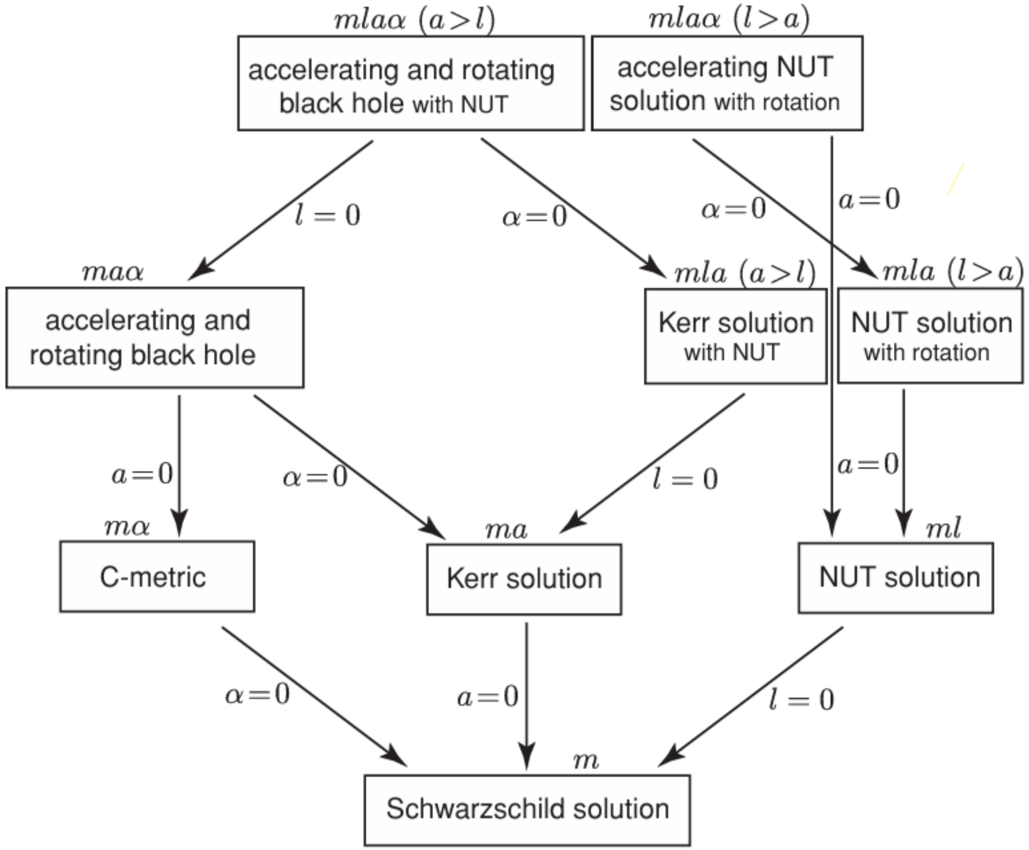
Introduction

- In **General Relativity** theory, the **principle action** is given by the **Einstein-Hilbert action** (plus boundary terms)

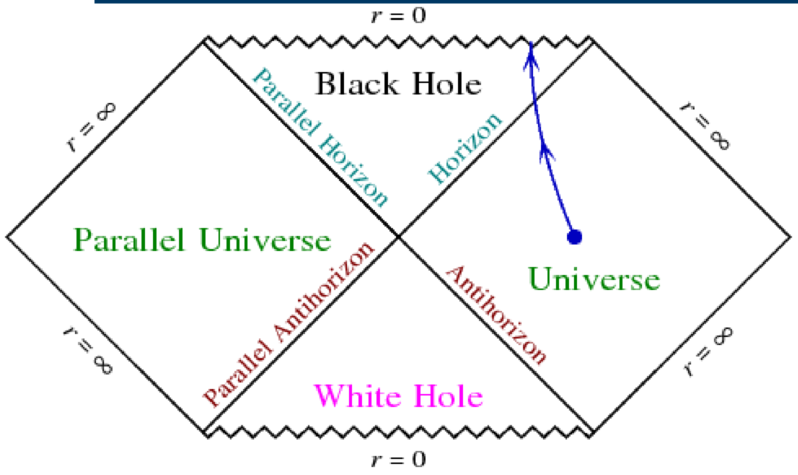
$$S_{EH}[g,\varphi] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(R - 2\Lambda + \mathcal{L}_M[g,\varphi]).$$

Here, $g_{\mu\nu}$ represents the metric tensor, $g := det(g_{\mu\nu})$, denotes the determinant of the metric tensor, $\kappa = 8\pi G/c^4$ where G is the Newton constant, and $\mathcal{L}_M$ is the lagrangian associated to matter, inside which φ represents the matter field. R is the Ricci scalar and Λ the cosmological constant.

- The **Schwarzschild solution**, the **static** and **spherically symmetric black hole in vacuum**, is part of the broader **Plebański-Demiański family** of solutions. This family encommpasses a wide range of spacetimes, including black holes, **accelerating** black holes, and other configurations with additional parameters like **electromagnetic charges**, **NUT charge**, **angular momentum**, and **cosmological constant.**

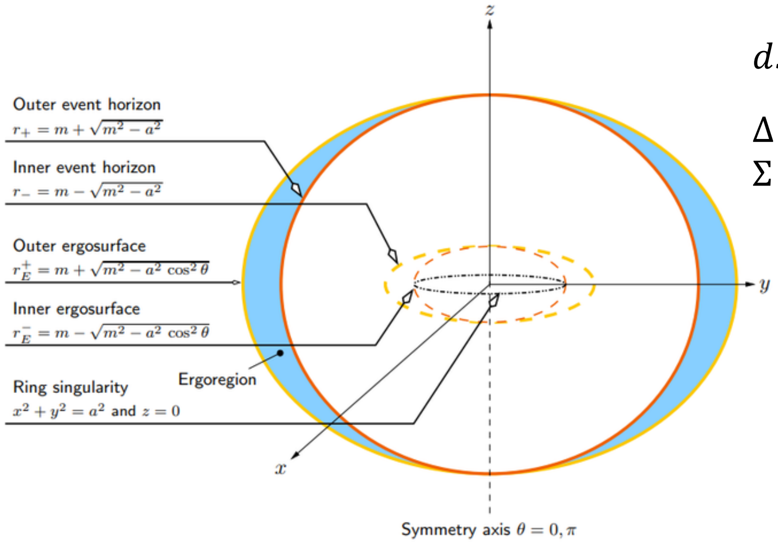


Two iconic solutions: Schwarzschild and Kerr black holes



$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- **Static** and **spherically symmetric**.
- Uniparametric.
- Gravitational field produced by a static source.



$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2$$

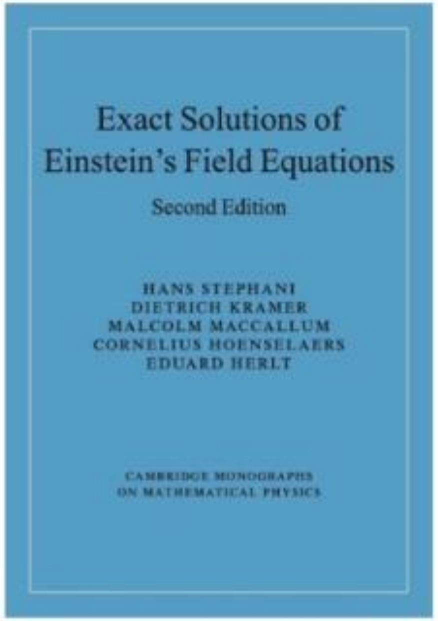
$$\Delta = r^2 - 2Mr + a^2$$

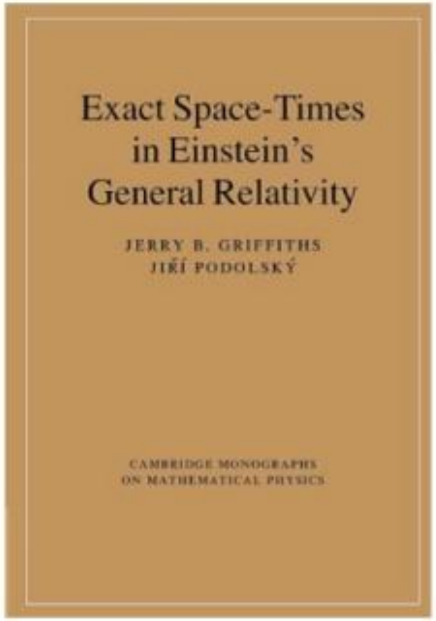
$$\Sigma = r^2 + a^2\cos^2\theta$$

- **Stationary** and **axially symmetric**.
- Biparametric.
- Gravitational field produced by a rotating source.
- **Ergoregions**.

- In **general relativity**, solving **Einstein’s equations** involves dealing with a set of **coupled, nonlinear partial differential equations.**

- The task of finding new **exact solutions** becomes increasingly challenging once the **simplest solutions** of the theory **are already well established**. For this reason, the use of tools such as **solution generating techniques** becomes particularly relevant, as they allow us to obtain solutions that would otherwise be extremely difficult to derive.





[1] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt, Exact solutions of Einstein’s field equations, Cambridge Monographs on Mathematical Physics (Cambridge Univ. Press, Cambridge, 2003).
[2] J. B. Griffiths and J. Podolsky, Exact Space-Times in Einstein’s General Relativity, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2009).

Ernst formalism

- The **Ernst scheme**[3,4], allow us to find a set of **Lie point symmetries** in the **Einstein-Maxwell theory**, whose **action** is given by

$$S[g_{\alpha\beta}, A_\mu] = \int d^4x\,\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right).$$

Where the **field equations** are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}\right),$$
$$\nabla_\mu F^{\mu\nu} = 0.$$

- Ernst showed that the Einstein-Maxwell equations can be rewritten in a sophisticated form.

- For a **stationary and axially symmetric spacetime**, the Einstein-Maxwell equations are equivalently represented by the **Ernst equations**

$$(Re(\boldsymbol{\varepsilon}) + |\Phi|^2)\nabla^2\boldsymbol{\varepsilon} = \vec{\nabla}\boldsymbol{\varepsilon}\cdot\left(\vec{\nabla}\boldsymbol{\varepsilon} + 2\Phi^*\vec{\nabla}\Phi\right),$$
$$(Re(\boldsymbol{\varepsilon}) + |\Phi|^2)\nabla^2\Phi = \vec{\nabla}\Phi\cdot\left(\vec{\nabla}\boldsymbol{\varepsilon} + 2\Phi^*\vec{\nabla}\Phi\right).$$

[3] F.J. Ernst, New formulation of the axially symmetric gravitational field problem, Phys. Rev. 167 (1968) 1175.
[4] F.J. Ernst, New Formulation of the Axially Symmetric Gravitational Field Problem. II, Phys. Rev. 168 (1968) 1415.

The **Ernst potentials** are defined as

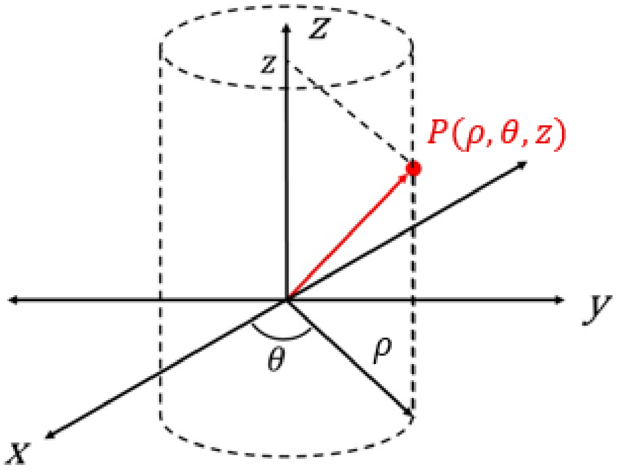
$$\boldsymbol{\varepsilon} = f - |\Phi|^2 + i\chi, \qquad \Phi = A_t + i\tilde{A}_\phi.$$

Ernst potentials are **complex scalar functions** constructed from the **Lewis-Weyl-Papapetrou metric** (the most general metric for a **stationary** and **axially symmetric** spacetime, which is **circular**), in cylindrical coordinates. In addition to a **stationary** and **axially symmetric gauge field**, the spacetime configuration reads

$$\begin{aligned} ds_e^2 &= -f(dt - \omega d\phi)^2 + f^{-1}[\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + dz^2)], \\ A_e &= A_t dt + A_\phi d\phi. \end{aligned}$$

Integrability conditions of the Einstein-Maxwell equations

$$\begin{aligned} \vec{e}_\phi \times \vec{\nabla}\tilde{A}_\phi &= \rho^{-1} f\big(\vec{\nabla}A_\phi + \omega\vec{\nabla}A_t\big), \\ \vec{e}_\phi \times \vec{\nabla}\chi &= -\rho^{-1} f^2 \vec{\nabla}\omega - 2\vec{e}_\phi \times Im\big(\Phi^* \vec{\nabla}\Phi\big). \end{aligned}$$



The Lewis-Weyl-Papapetrou metric **does not represent the only ansatz** for a stationary and axially symmetric spacetime. An **inequivalent** ansatz is given by considering a **double Wick rotation** $t \to i\hat{\phi}$ & $\phi \to i\hat{t}$. Renaming $\hat{\mathrm{t}} = t$ & $\hat{\phi} = \phi$

$$
\begin{aligned}
ds_m^2 &= f(d\phi - \omega dt)^2 + f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) - \rho^2 dt^2], \\
A_m &= A_t dt + A_\phi d\phi.
\end{aligned}
$$

Now with a **redefinition** of the **complex potentials**

$$\varepsilon = -f - |\Phi|^2 - i\chi, \qquad \Phi = A_\phi + i\tilde{A}_t.$$

Additionally, one of the **integrability conditions change**

$$\vec{e}_\phi \times \vec{\nabla}\tilde{A}_t = \rho^{-1} f\left(\vec{\nabla}A_t + \omega\vec{\nabla}A_\phi\right).$$

The other differential equation **remains the same**.

The Kinnersley group

The **main feature** of the Ernst equations is that they contain a set of **Lie point symmetries** given by

$$
\begin{aligned}
G_1[a]: &\quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + ia, && \Phi = \Phi_0,\\
G_2[\alpha]: &\quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 - 2\alpha^*\Phi_0 - |\alpha|^2, && \Phi = \Phi_0 + \alpha,\\
D[\lambda]: &\quad \boldsymbol{\varepsilon} = |\lambda|^2\boldsymbol{\varepsilon}_0, && \Phi = \lambda\Phi_0,\\
E[c]: &\quad \boldsymbol{\varepsilon} = \frac{\boldsymbol{\varepsilon}_0}{1 + ic\boldsymbol{\varepsilon}_0}, && \Phi = \frac{\Phi_0}{1 + ic\boldsymbol{\varepsilon}_0},\\
H[\beta]: &\quad \boldsymbol{\varepsilon} = \frac{\boldsymbol{\varepsilon}_0}{1 - 2\beta^*\Phi_0 - |\beta|^2\boldsymbol{\varepsilon}_0}, && \Phi = \frac{\beta\boldsymbol{\varepsilon}_0 + \Phi_0}{1 - 2\beta^*\Phi_0 - |\beta|^2\boldsymbol{\varepsilon}_0}.
\end{aligned}
$$

Where α, β y λ are complex parameters, while b and j are real. The transformations G_1 and G_2 are **gauge transformations** that transform the potentials but leave the metric and the gauge field invariant. D is a **duality-rescaling transformation**, E denotes the **Ehlers transformations**, and H represents the **Harrison transformations.**

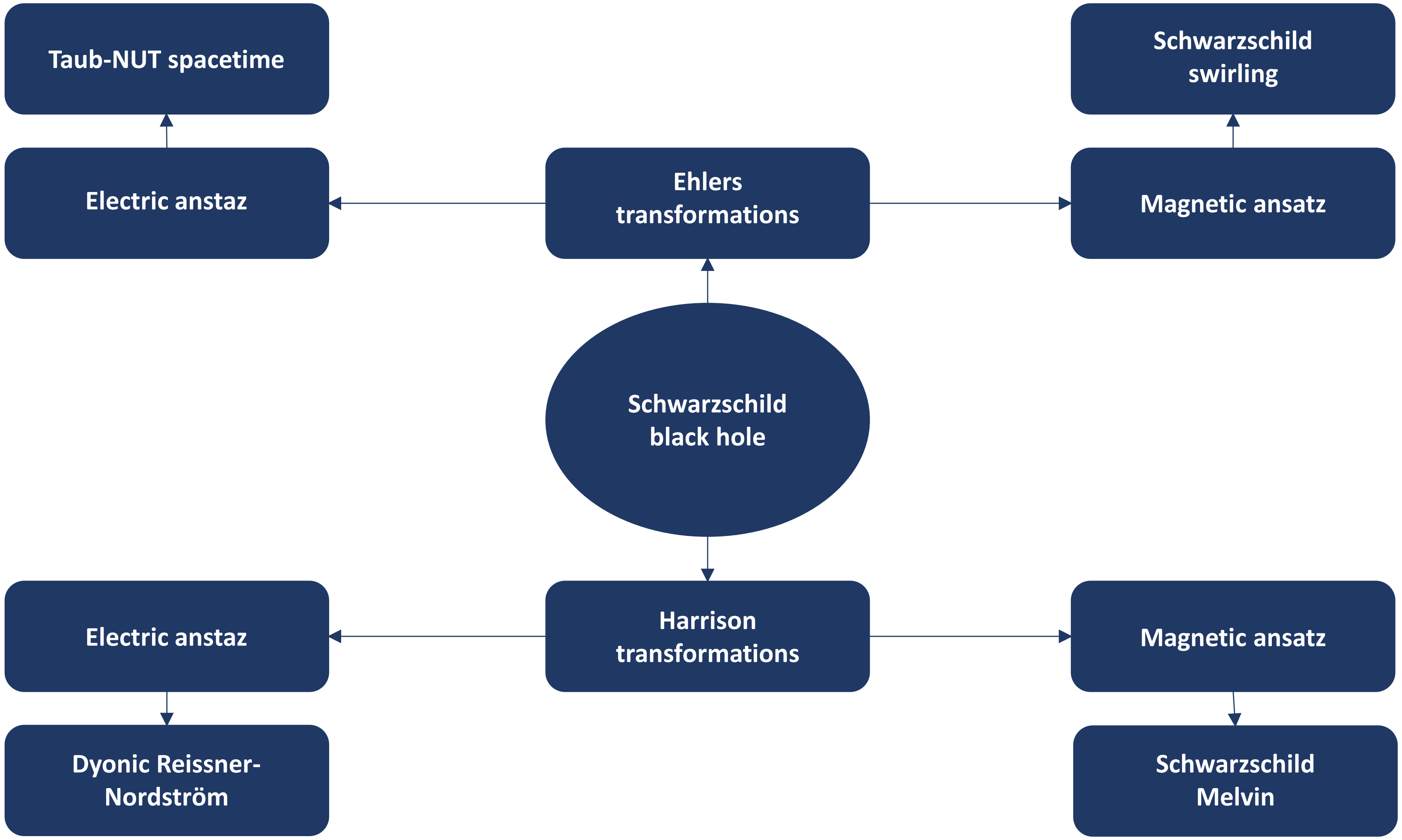
There exists **another transformation**, which is given by the **composition** of G_1, D and E

$$
I: \quad \boldsymbol{\varepsilon} = \frac{1}{\boldsymbol{\varepsilon}_0}, \qquad \Phi = \frac{\Phi_0}{\boldsymbol{\varepsilon}_0}.
$$

This is the **inversion map**[5], a **discrete symmetry of the Ernst equations.**

[5] J. Barrientos, A. Cisterna, M. Hassaine, K. Muller and K. Pallikaris, [arXiv:2506.07166 [gr-qc]].

The choice on the ansatz is crucial, as it acts differently on a seed solution. In the **electric ansatz**, the (electromagnetic/NUT) charge remains localized within the black hole. In contrast, the **magnetic ansatz** carries the charge to infinity, altering the **asymptotic behavior** of the solution at infinity.



Mixing “Magnetic” and “Electric” Ehlers-Harrison transformations

- The work **arXiv:2401.02924 [gr-qc]** was done in collaboration with: José Barrientos, Adolfo Cisterna, Ivan Kolar, Marcelo Oyarzo, Konstantinos Pallikaris.

- This work can be summarized in **two main results:**
  - The **construction** of the **Melvin-Bonnor-swirling spacetime.** A spacetime that combines the swirling background and the spacetime containing an external electromagnetic field in the background.
  - The composition of the Ehlers and Harrison transformations acting on the different LWP ansatz, obtaining **four novel stationary and axisymmetric asymptotically non-flat type I spacetimes.**

Eur. Phys. J. C (2024) 84:724
https://doi.org/10.1140/epjc/s10052-024-13093-x
THE EUROPEAN PHYSICAL JOURNAL C
Regular Article - Theoretical Physics

Mixing “Magnetic” and “Electric” Ehlers–Harrison transformations: the electromagnetic swirling spacetime and novel type I backgrounds

José Barrientos[1,2,a], Adolfo Cisterna[1,3,b], Ivan Kolář[3,c], Keanu Müller[4,d], Marcelo Oyarzo[4,e], Konstantinos Pallikaris[5,f]

[1] Sede Esmeralda, Universidad de Tarapacá, Avenida Luis Emilio Recabarren 2477, Iquique, Chile
[2] Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic
[3] Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic
[4] Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile
[5] Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, W. Ostwaldi 1, 50411 Tartu, Estonia

Received: 6 May 2024 / Accepted: 6 July 2024
© The Author(s) 2024

Abstract In this paper, we obtain a complete list of stationary and axisymmetric spacetimes, generated from a Minkowski spacetime using the Ernst technique. We do so by operating on the associated seed potentials with a composition of Ehlers and Harrison transformations. In particular, assigning an additional “electric” or “magnetic” tag to the transformations, we investigate the new spacetimes obtained either via a composition of magnetic Ehlers and Harrison transformations (first part) or via a magnetic-electric combination (second part). In the first part, the resulting type D spacetime, dubbed electromagnetic swirling universe, features key properties, separately found in swirling and (Bonnor–)Melvin spacetimes, the latter recovered in appropriate limits. A detailed analysis of the geometry is included, and subtle issues are addressed. A detailed proof that the spacetime belongs to the Kundt family, is included, and a notable relation to the5 planar-Reissner–Nordström-NUT black hole is also meticulously worked out. This relation is further exploited to reverse-engineer the form of the solution in the presence of a nontrivial cosmological constant. A Schwarzschild black hole embedded into the new background is also discussed. In the second part, we present four novel stationary and axisymmetric asymptotically nonflat type I spacetimes, which are naively expected to be extensions of the Melvin or swirling solution including a NUT parameter or electromagnetic charges. We actually find that they are, under conditions, free of curvature and topological singularities, with the physical meaning of the electric transformation parameters in these backgrounds requiring further investigation.

[a] e-mail: jbarrientos@academicos.uta.cl (corresponding author)
[b] e-mail: adolfo.cisterna@mff.cuni.cz
[c] e-mail: ivan.kolar@matfyz.cuni.cz
[d] e-mail: keanumuller2016@udec.cl
[e] e-mail: moyarzo2016@udec.cl
[f] e-mail: konstantinos.pallikaris@ut.ee

Contents

Published online: 23 July 2024
Springer

The magnetic Universe

- Also known as the **Bonnor-Melvin solution**, was first found by **Bonnor**[8] (1954), and it was later rediscovered by **Melvin**[9] (1966).

- It describes a **static** and **axially symmetric magnetic field** immersed in its own gravitational field.

- **The solution** of the **Einstein-Maxwell** field equations in **cylindrical coordinates** reads

$$ds_{EM}^2 = \frac{\rho^2}{V^2} d\phi^2 + V^2(-dt^2 + d\rho^2 + dz^2),$$

where $V(\rho) \coloneqq 1 + X\bar{X}\rho^2$, and $\bar{X}$ is the complex conjugate of $X \coloneqq (E + iB)/2$, with E the intensity of the **electric field**, and B the **magnetic field.**

- This metric is accompanied by the **gauge field**

$$A = -zEdt - \frac{B\rho^2}{2V} d\phi.$$

- By fixing $E = B = 0$ we recover the **Minkowski spacetime**.

[8] W.B. Bonnor, Static magnetic fields in general relativity, Proceedings of the Physical Society. Section A 67 (1954) 225.
[9] M.A. Melvin and J.S. Wallingford, Orbits in a magnetic universe, Journal of Mathematical Physics 7 (1966) 333.

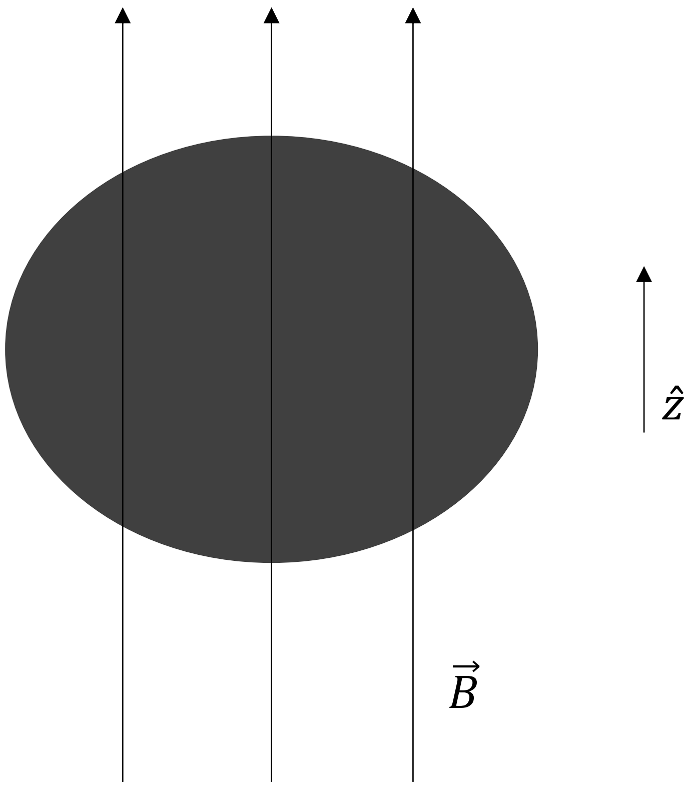
- The electromagnetic universe **can be obtained from** a **Minkowski** seed via a **magnetic Harrison transformation**, fixing the parameter $\alpha = i\bar{X}$.

- The Bonnor-Melvin spacetime is of **type D.**

- It is possible to **embed** a black hole in this spacetime.

- The embedding is of **type I.**

- This solution can be obtained as a certain **limit** of the **Reissner–Nordström** solution or the **charged C-metric**. In the **first case**, it is obtained after "**planarize**" the solution, performing a double Wick rotation in t and ϕ, and then applying coordinate transformations and reparametrizations. In the **second case**, it is obtained after "**moving**" the **black holes to infinity**[10].



[10] L. Havrdova and P. Krtous, Melvin universe as a limit of the C-metric, Gen. Rel. Grav. 39 (2007) 291–296, [gr-qc/0611092].

Swirling Universe

- On the other hand, another solution of the Einstein field equations is the **swirling spactime**[11]**.** This is a **stationary vacuum solution**, which in **cylindrical coordinates** reads

$$ds^2 = \frac{\rho^2}{1 + j^2\rho^4}(d\phi + 4jzdt)^2 + (1 + j^2\rho^4)(-dt^2 + d\rho^2 + dz^2),$$

- The solution describes a **rotating spacetime.**

- The metric function ω grows infinitely large as $|z| \to \infty$. Being linear in z, it is constant on fixed- z planes and zero on the equatorial plane $z = 0$, where it changes sign.

- By fixing $j = 0$ we recover the **Minkowski spacetime.**

[11] M. Astorino, R. Martelli and A. Vigano, Black holes in a swirling universe, Phys. Rev. D 106 (2022) 064014 [2205.13548].

- This solution **can be obtained from** a **Minkowski** seed via a **magnetic Ehlers transformation** with real parameter j.

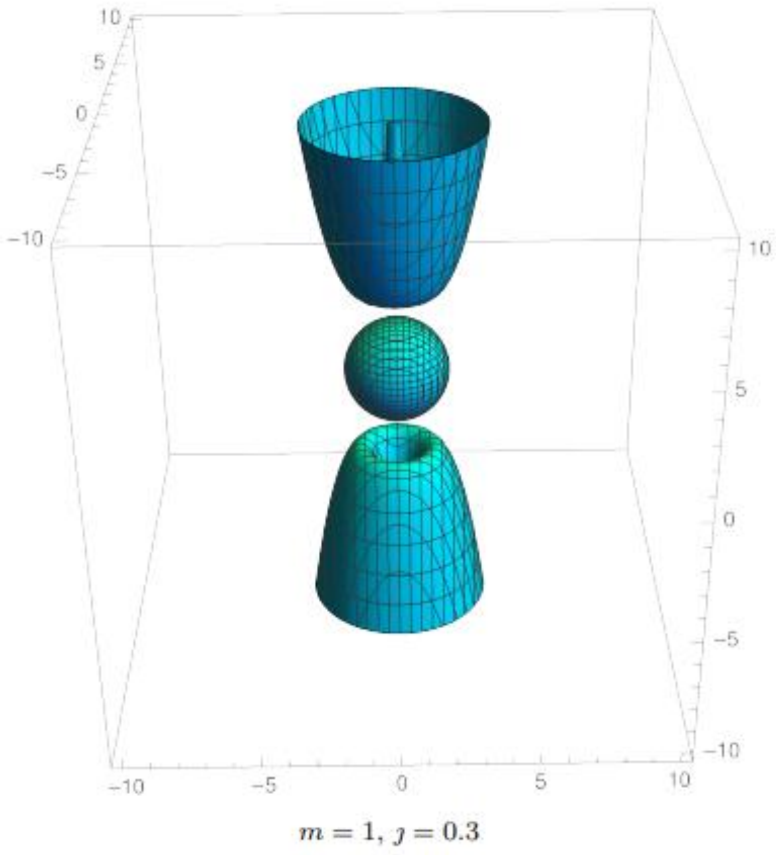
- The swirling Universe is of **type D.**

- It is possible to **embed** a black hole in this spacetime.

- The embedding is of **type I.**

- Represents a stationary and axially symmetric black hole, that rotates due to the dragging effect of the background. Because of this there are ergoregions like in the Kerr black hole case, but in this case the ergoregions extend to infinity along the axis of symmetry.

- This solution can be obtained as a certain **limit** of the **Taub-NUT** solution. In this case, it is obtained after "**planarize**" the solution, performing a double Wick rotation in $t$and ϕ, and then applying coordinate transformations and reparametrizations.



Ergoregions for the Schwarzschild black hole embedded in a rotating background[11].

[11] M. Astorino, R. Martelli and A. Vigano, Black holes in a swirling universe, Phys. Rev. D 106 (2022) 064014 [2205.13548].

The electromagnetic swirling Universe

- This solution is obtained by considering the **Minkowski spacetime** in cylindrical coordinates as a seed

$$ds_0^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi^2,$$

via a **composition of magnetic Ehlers and Harrison transformations**

$$E_m[j] \circ H_m[i\bar{X}],$$

where the complex parameter $X \coloneqq (E + iB)/2$.

- The **electromagnetic swirling universe** (EMS) solution reads

$$ds_{EMS}^2 = \frac{\rho^2}{V^2 + j^2\rho^4}(d\phi + 4jzdt)^2 + (V^2 + j^2\rho^4)(-dt^2 + d\rho^2 + dz^2),$$

$$A = \frac{\rho^2(2z[EV^2\rho^{-2} + j(2BV - jE\rho^2)]dt + (BV - jE\rho^2)d\phi)}{2(V^2 + j^2\rho^4)}.$$

- There are **no event horizons.**
- Absence of **conical singularities.**
- Absence of **Misner string.**
- Free of **Closed Timelike Curves** (CTCs).
- Free of **curvature singularities.**
- The EMS spacetime is of **type D.**
- The **Schwarzschild** embedding is of Petrov **type I.**
- The EMS spacetime is **not asymptotic to the swirling spacetime**, since the gauge field does not vanish as $\rho \to \infty$.

- The electric and magnetic fields in the EMS universe reads

$$\boldsymbol{E} = -\frac{2jBV\rho^2 + E(V^2 - j^2\rho^4)}{(V^2 + j^2\rho^4)^2}\hat{z},$$
$$\boldsymbol{B} = (\boldsymbol{E})_{(E,B)\to(-B,E)}.$$

Both depend only on ρ with field lines parallel to the axis of symmetry, in the vicinity of which they acquire a constant profile, $-E\hat{z}$ and $B\hat{z}$, respectively.

- For $X = 0$ we recover the swirling metric.
- For $j = 0$ the resulting spacetime is the electromagnetic Universe.

Mixing Magnetic and Electric Ansatze

- The composition of transformations raises the question: What happens when combining transformations that act on different LWP ansatz?

- Essentially, we want to ask whether starting from the Minkowski metric as a seed, the composition of transformations within the different ansatz leads to new background geometries free of singularities, topological defects, and other pathologies.

- When the seed is Minkowski, the number of admissible transformation combinations is reduced. For instance, applying any electric-type transformation still yields Minkowski spacetime, since the corresponding charge (electromagnetic or NUT) cannot be supported by the geometry.

Electromagnetic universe and electric Harrison transformations

- This spacetimes is obtaned from the composition of $H_e[Q] \circ H_m[i\bar{X}]$. The solution reads

$$ds^2 = -f(dt - \omega d\phi)^2 + f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2],$$

with

$$f = \frac{V^2}{H^2 + 2H(q_e E - q_m B)z + 16|QX|^2 z^2}, \omega = \frac{(q_e B + q_m E)\big(1 - |Q|^2(V + 4|X|^4\rho^2 z^2)\big)\rho^2}{V}, e^{2\gamma} = V^4.$$

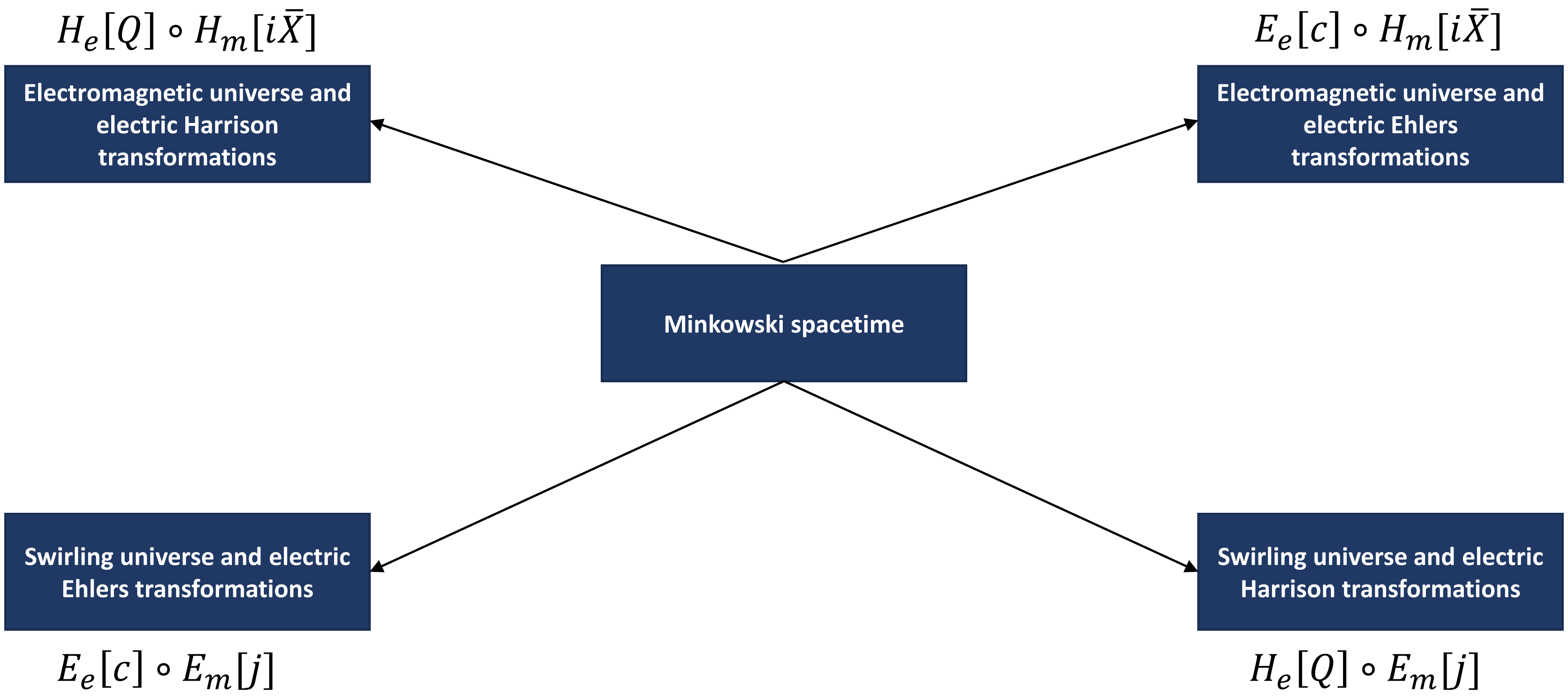
Where we defined $H(\rho, z) \coloneqq 1 - |Q|^2\,(V^2 - 4|X|^2 z^2)$, with $Q \coloneqq (q_e + iq_m)/2$.

The spacetime corresponds to a **stationary** background which is **not asymptotically flat** with $|\mathrm{Q}| > 1$ and $q_m E \neq -q_e B$.
In the Petrov classification, it is of **type I**. In the limit where the electromagnetic field vanishes $X \to 0$, the spacetime reduces to Minkowski. In the limit where the charges go to zero $Q \to 0$, we recover the electromagnetic universe.
The solution **exhibits** closed timelike curves (CTCs). These regions can be **removed** by appropriately tuning the parameters as $q_m E = -q_e B$; however, in this limit the **rotation $\boldsymbol{\omega}$ vanishes** and the spacetime develops a **naked singularity.**

$$A_t = \frac{f\left((V + (q_e E - q_m B)z)\left(-Ez + \frac{q_e \epsilon_0}{2}\right) - \left(Bz + \frac{q_m \epsilon_0}{2}\right)(q_e B + q_m E)z\right)}{V^2}$$

$$A_\phi = \frac{\rho^2\left(4|X|^2 q_m z + (V + 4|X|^2 z^2)\left(\frac{(3q_e^2 - q_m^2)B}{4} + q_e q_m E\right) - B\right)}{2V} - \omega A_t.$$

- Finally, there are four spacetimes that do not commute, if the seed is Minkowski.



Conclusions

- By employing the Lie point symmetries of the Ernst equations, we obtained a new type D background solution, the electromagnetic swirling universe, which corresponds to the combination of the swirling spacetime and the electromagnetic spacetime, both of type D. This was achieved by composing a Harrison transformations and an Ehlers transformation, both applied in the same magnetic ansatz, taking Minkowski as a seed.

- On the other hand, through the composition of the Ehlers and Harrison transformations applied to the electric and magnetic ansatze, we derived from Minkowski spacetime, four novel type I background solutions, representing stationary and axially symmetric spacetimes.

Thanks