Rotating spacetimes with scalar hair in four and five dimensions

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José Barrientos

Universidad de Tarapacá
Institute of Mathematics, Czech Academy of Sciences

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UNIVERSIDAD DE TARAPACÁ
Universidad del Estado
Sede Iquique

Agencia Nacional de Investigación y Desarrollo
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Gobierno de Chile

A general line element possessing both, timelike and axial symmetries, ∂_t and ∂_φ, can be expressed in the LWP form[1]

$$ds^2 = -e^{2U}[dt - \omega d\varphi]^2 + e^{-2U}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right], \qquad (1)$$

where U, ω and γ are functions of (ρ, z). In the static case, the vacuum field equations $R_{\mu\nu} = 0$ imply

$$U_{,\rho\rho} + \frac{1}{\rho}U_{,\rho} + U_{,zz} = 0 = \nabla^2_{\mathbb{E}^3}U, \qquad (2)$$

and the quadratures

$$\gamma_{,\rho} = \rho(U^2_{,\rho} + U^2_{,z}), \qquad \gamma_{,z} = 2\rho U_{,\rho}U_{,z}. \qquad (3)$$

For any solution of U, the function γ can be obtained by integrating the quadratures above.

[1] *Exact solutions of Einstein's field equations*, H. Stephani *et. al*

In spherical coordinates (r, θ), the general solution to $\nabla^2_{\mathbb{E}^3} U = 0$ corresponds to taking the multipolar expansion on $\mathbb{E}^3$,

$$U = \frac{a_0}{r} + \sum_{\ell=1}^{\infty} \left(\frac{a_\ell}{r^{\ell+1}} + b_\ell r^\ell \right) P_\ell(\cos\theta). \tag{4}$$

Then,

$$\begin{aligned}\gamma = \sum_{\ell,m=0}^{\infty} \left[\frac{(\ell+1)(m+1)a_\ell a_m}{(\ell+m+2) r^{\ell+m+2}} \left(P_{\ell+1} P_{m+1} - P_\ell P_m \right) \right] \\ + \sum_{\ell,m=1}^{\infty} \left[\frac{\ell m b_\ell b_m r^{\ell+m}}{\ell+m} \left(P_\ell P_m - P_{\ell-1} P_{m-1} \right) \right].\end{aligned} \tag{5}$$

- Consequently, a general family of vacuum solutions, determined by the parameters (a_ℓ, b_ℓ) is obtained
- In this sense, all the vacuum solutions of this class are formally (mathematically) known
- In the linear approximation, $g_{tt} \sim -1 - 2U$, and U satisfies the Laplace equation. Then, it is natural to regard U as the analogue of a Newtonian potential

Zipoy-Voorhees spacetime

- Corresponds to the Newtonian potential of a finite rod of mass M and length 2ℓ, whose mass per unit length is arbitrary $\sigma = M/2\ell$

$$U(\rho, z) = \sigma \ln \left[\frac{r_+ + r_- - 2M}{r_+ + r_- + 2M} \right], \; r_\pm^2 \equiv \rho^2 + (z \pm M)^2.$$

- Besides the mass M of the corresponding rod, it is usually described by the parameter δ, which is known as the deformation parameter and is defined such that $\delta = 2\sigma = M/\ell$

$$ds^2_{\mathrm{ZV}} = -f^\delta dt^2 + \frac{\left[\left(\frac{f}{g}\right)^{\delta^2} g \left(\frac{dr^2}{f} + r^2 d\theta^2 \right) + f r^2 \sin^2\theta d\varphi^2 \right]}{f^\delta}, \quad (6)$$

with

$$f = \left(1 - \frac{2M}{r}\right), \quad g = \left(1 - \frac{2M}{r} + \frac{M^2 \sin^2\theta}{r^2}\right). \quad (7)$$

- When $\delta = 1$, the solution reduces to Schwarzschild

Including a matter source

- When considering matter fields, such as in electro-vacuum, the situation is slightly different, nonlinearities appear in the equation for U. Via the Ernst scheme, sophisticated geometries can be constructed (Keanu's talk!)

- Beyond that, black holes with scalar hair have been studied since the early '60s

- Several no-hair theorems demonstrate that GR typically cannot support scalar hair on regular, asymptotically flat black hole configurations.[2] Free scalar fields exhibit singular behavior, making the event horizon also singular

- However, these theorems depend on the specific theory being considered[3]

[2] J. Chase, 1970. J. Bekenstein, 1972

[3] C. Herdeiro and E. Radu, 2015

- Via disformal transformations of the form $\bar{g}_{\mu\nu} = C g_{\mu\nu} + D\partial_\mu\Phi\partial_\nu\Phi$, the Einstein-Scalar system is known to be connected with all well-posed families of scalar-tensor theories... Horndeski, DHOST[4]

- A typical example of this is the BBMB black hole,[5] a spacetime arising in the Einstein-Conformal-Scalar system

$$I = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[R - \kappa\partial_\mu\Phi\partial^\mu\Phi\right], \tag{8}$$

$$\bar{g}_{\mu\nu} = \cosh^2\left(\sqrt{\frac{\kappa}{6}}\Phi\right) g_{\mu\nu}, \quad \Psi = \sqrt{\frac{6}{\kappa}}\tanh\left(\sqrt{\frac{\kappa}{6}}\Phi\right), \tag{9}$$

we land in the following theory

$$\bar{I} = \frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}\left[\bar{R} - \kappa\left(\partial_\mu\Psi\partial^\mu\Psi + \frac{1}{6}\bar{R}\Psi^2\right)\right], \tag{10}$$

$$d\bar{s}^2 = -\left(1 - \frac{M}{r}\right)^2 + \frac{dr^2}{(1-\frac{M}{r})^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad \Psi = \sqrt{\frac{6}{\kappa}}\frac{M}{r-M}.$$

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[4] D. Langlois and K. Noui, 2015

[5] N. Bocharova, K. Bronnikov and V. Melnikov, 1970.

- In the conformal frame, the scalar sector's symmetry simplifies the integration of field equations, allowing solutions with charges, NUT parameters, and acceleration.[6] Rotation is harder to include

- In the Einstein frame, integration is more complex. However, its simpler Lagrangian permits certain theorems that admit scalar hair, often leading to singular solutions consistent with no-hair theorems

- As a first glance, Buchdahl's theorem[7] exemplifies this, acting as a Lie point symmetry that lets one "dress" static vacuum spacetimes with a scalar field

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[6] A. Anabalon and H. Maeda, 2009. C. Charmousis *el al*, 2009, 2013

[7] H. Buchdahl, 1956, 1959

Buchdahl transformation: given a static vacuum solution of the form,

$$ds_0^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{aa}(x^k)(dx^a)^2 + g_{ij}(x^k)dx^i dx^j, \qquad (11)$$

which we say static with respect to a coordinate a, that is

$$g_{ai} = 0 = \partial_a g_{\mu\nu},$$

the spacetime configuration

$$\begin{aligned} ds_{ES}^2 &= (g_{aa})^\beta (dx^a)^2 + (g_{aa})^{\frac{1-\beta}{d-3}} g_{ij} dx^i dx^j, \\ \Phi &= \sqrt{\frac{(d-2)(1-\beta^2)}{4(d-3)\kappa}} \ln(g_{aa}), \qquad |\beta| < 1, \end{aligned} \qquad (12)$$

will correspond to a solution of the Einstein-Scalar field equations

$$R_{\mu\nu} = \kappa \partial_\mu \Phi \partial_\nu \Phi, \qquad \Box \Phi = 0. \qquad (13)$$

Well known example: FJNW solution

Schwarzschild as seed:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2.$$

- With respect to ∂_t

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right)^{\beta} dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^{\beta}} + \left(1 - \frac{2M}{r}\right)^{1-\beta} r^2 d\Omega^2, \\ \Phi &= \sqrt{\frac{1-\beta^2}{2\kappa}} \ln\left(1 - \frac{2M}{r}\right). \end{aligned} \tag{14}$$

- Fisher-Janis-Newman-Winicour solution

- Singular for $r = 2M$

$$R \propto \frac{1}{r^{2-\beta}(r-2M)^{2-\beta}}. \tag{15}$$

Myers-Perry black holes *à la* Buchdahl

- One can show that for the Myers-Perry solutions, the hypothesis allowing the Buchdahl transformation is not respected unless (and at least) one of the rotation parameters vanishes (cyclic coordinate)

- In five dimensions, the Myers-Perry solution with two rotations has the form

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2Mr^2}{S(r,\theta)}\right)dt^2+\frac{4Mr^2(a\sin^2\theta d\xi+b\cos^2\theta d\chi)}{S(r,\theta)}dt \\
&+\sin^2\theta\left(r^2+a^2+\frac{2Mr^2a^2\sin^2\theta}{S(r,\theta)}\right)d\xi^2+\cos^2\theta\left(r^2+b^2+\frac{2Mr^2b^2\cos^2\theta}{S(r,\theta)}\right)d\chi^2 \\
&+\frac{4Mr^2ab\sin^2\theta\cos^2\theta}{S(r,\theta)}d\xi d\chi+\frac{S(r,\theta)}{\Delta_M(r)}\left[dr^2+\frac{\Delta_M(r)}{r^2}d\theta^2\right].
\end{aligned}
\tag{16}
$$

- If $b=0$, then ∂_χ is the corresponding Killing vector. A rotating five-dimensional spacetime with scalar hair also exists

What about stationary cases?

Rotating spacetimes with a free scalar field

- Eriș and Gürses have already answered the question[8]

- Any vacuum solution in the LWP form,

$$ds^2 = -e^{2U}[dt - \omega d\varphi]^2 + e^{-2U}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right],$$

  can be extended to a corresponding solution of the Einstein-Scalar system $(U, \omega, \gamma_{\text{new}} = \gamma + \gamma^{\Phi}, \Phi)$, provided γ^{Φ} and Φ satisfy the following equations

$$\gamma^{\Phi}_{,\rho} = \rho\left(\Phi^2_{,\rho} - \Phi^2_{,z}\right), \quad \gamma^{\Phi}_{,z} = 2\rho\Phi_{,\rho}\Phi_{,z}, \quad \nabla^2_{\mathbb{E}^3}\Phi = 0. \qquad (17)$$

- Once again, the new nonlinear function is integrated via the quadratures (17), after a solution of the Laplace equation for the scalar field Φ is selected

- The use of cylindrical coordinates complicates the analysis of rotating solutions

[8] A. Eris and M. Gurses, 1977

- In fact, using the following spherical-like ansatz for the metric

$$ds^2 = -e^{2U}(dt-\omega d\varphi)^2+e^{-2U}\left(e^{2\gamma}\left[\frac{dr^2}{\Delta_1}+\frac{d\theta^2}{\Delta_2}\right]+\varrho^2 d\varphi^2\right), \quad (18)$$

where the functions U, ω, γ, ϱ^2 depend on (r,θ) and, where $\Delta_1 = \Delta_1(r)$ and $\Delta_2 = \Delta_2(\theta)$, the scalar field backreaction is described by a solution to the system

$$\begin{aligned} \left(\ln\varrho^2\right)_{,r}\gamma^{\Phi}_{,\theta} + \left(\ln\varrho^2\right)_{,\theta}\gamma^{\Phi}_{,r} &= 4\Phi_{,r}\Phi_{,\theta}, \\ \Delta_1\left(\ln\varrho^2\right)_{,r}\gamma^{\Phi}_{,r} - \Delta_2\left(\ln\varrho^2\right)_{,\theta}\gamma^{\Phi}_{,\theta} &= 2(\Delta_1\Phi^2_{,r} - \Delta_2\Phi^2_{,\theta}). \end{aligned} \quad (19)$$

- In these coordinates, the quadratures depend on the seed metric, and Φ obeys the curved Klein-Gordon equation

- A full solution is not ensured, but is solvable in several cases

Kerr-Newman-NUT configuration with scalar hair and solutions with $\Phi = \Phi(r,\theta)$

- A Kerr-Newman-NUT generalization can be obtained, yielding

$$ds^2 = -\frac{\Delta(r)}{S(r,\theta)}\left(dt - (a\sin^2\theta + 2l\cos\theta)d\varphi\right)^2 + \frac{\sin^2\theta}{S(r,\theta)}\left(a dt - (r^2 + a^2 + l^2)d\varphi\right)^2 + S(r,\theta)H(r,\theta)\left(\frac{dr^2}{\Delta(r)} + d\theta^2\right)$$

$$A = -\frac{er}{S(r,\theta)}dt + \frac{er(a\sin^2\theta + 2l\cos\theta)}{S(r,\theta)}d\varphi, \tag{20}$$

with

$$\Delta(r) = r^2 - 2Mr + a^2 - l^2 + e^2, \quad S(r,\theta) = r^2 + (l - a\cos\theta)^2, \tag{21}$$

and where the scalar field backreaction is given by

$$
\begin{aligned}
\Phi(r,\theta) = & \frac{\Sigma}{2\sqrt{M^2+l^2-a^2-e^2}} \ln\left(\frac{r-M-\sqrt{M^2+l^2-a^2-e^2}}{r-M+\sqrt{M^2+l^2-a^2-e^2}}\right) \\
& + \frac{\Theta}{2\sqrt{M^2+l^2-a^2-e^2}} \ln\left(\frac{1-\cos\theta}{1+\cos\theta}\right), \\
H(r,\theta) = & \left(1+\frac{M^2+l^2-a^2-e^2}{\Delta(r)}\sin^2\theta\right)^{-\frac{\Sigma^2}{M^2+l^2-a^2-e^2}} \\
& \times \left(M^2+l^2-a^2-e^2+\frac{\Delta(r)}{\sin^2\theta}\right)^{-\frac{\Theta^2}{M^2+l^2-a^2-e^2}} \\
& \times \left(\frac{r-M-\sqrt{M^2+l^2-a^2-e^2}\cos\theta}{r-M+\sqrt{M^2+l^2-a^2-e^2}\cos\theta}\right)^{\frac{2\Sigma\Theta}{M^2+l^2-a^2-e^2}} .
\end{aligned}
\tag{22}
$$

Here, Σ, Θ are integration constants that represent the scalar hair.

- The geometry described by the metric is asymptotically flat, possesses a naked singularity at the would-be horizon $\Delta_M(r) = 0$ and has the Kerr ring singularity located at $S(r, \theta) = 0$

- The singular behavior can be seen from the Ricci scalar ($\Theta = l = 0$)

$$R = \frac{2\Sigma^2}{\Delta_M(r) S(r,\theta)} \left(1 + \frac{M^2 - a^2}{\Delta_M(r)} \sin^2\theta\right)^{\frac{\Sigma^2}{M^2 - a^2}} . \tag{23}$$

- The static limit $a = 0$ reproduces a ZV-FJNW spacetime with a fixed value of the ZV-deformation parameter ($\delta\beta = 1$, $1 - \delta^2 = -\Sigma^2/M^2$)

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right) dt^2 + \left(\frac{1 - \frac{2M}{r} + \frac{M^2 \sin^2\theta}{r^2}}{1 - \frac{2M}{r}}\right)^{-\frac{\Sigma^2}{M^2}} \left(\frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\theta^2\right) \\ &\quad + r^2 \sin^2\theta d\varphi^2, \\ \Phi(r) &= \frac{\Sigma}{2M} \ln\left(1 - \frac{2M}{r}\right) . \end{aligned} \tag{24}$$

- Higher-dimensional extensions to the Weyl problem have been explored,[9] but not for the Einstein-Scalar theory
- One might attempt to extend the Eriş-Gürses theorem to higher dimensions, but working with the Myers-Perry solution in cylindrical coordinates would significantly complicate the calculations
- We have seen that in four dimensions, a conformal transformation of the non-Killing sector of the vacuum solution using Boyer-Lindquist coordinates allows for a simple integration of the problem

$$\begin{aligned} ds^2 = & -\left(1-\frac{2Mr^2}{S(r,\theta)}\right)dt^2 + \frac{4Mr^2(a\sin^2\theta d\xi + b\cos^2\theta d\chi)}{S(r,\theta)}dt \\ & + \sin^2\theta\left(r^2+a^2+\frac{2Mr^2a^2\sin^2\theta}{S(r,\theta)}\right)d\xi^2 + \cos^2\theta\left(r^2+b^2+\frac{2Mr^2b^2\cos^2\theta}{S(r,\theta)}\right)d\chi^2 \\ & + \frac{4Mr^2ab\sin^2\theta\cos^2\theta}{S(r,\theta)}d\xi d\chi + \frac{H(r,\theta)S(r,\theta)}{\Delta_M(r)}\left[dr^2+\frac{\Delta_M(r)}{r^2}d\theta^2\right], \end{aligned} \tag{25}$$

with

$$\begin{aligned} S(r,\theta) &= r^2\left(r^2+a^2\cos^2\theta+b^2\sin^2\theta\right), \\ \Delta_M(r) &= (r^2+a^2)(r^2+b^2)-2Mr^2. \end{aligned} \tag{26}$$

[9] R. Emparan and H. Reall, 2002. T. Harmark, 2004. C. Charmousis and R. Gregory, 2004

$$H(r,\theta) = \left(4\left[1 + \frac{\left(M - \frac{(a+b)^2}{2}\right)\left(M - \frac{(a-b)^2}{2}\right)}{\Delta_M(r)}\sin^2(2\theta)\right]\right)^{-\frac{\Sigma^2}{(2M-|a^2-b^2|)^2}}$$

$$\times\left(4\left[\frac{\Delta_M(r)}{\sin^2(2\theta)} + \left(M - \frac{(a+b)^2}{2}\right)\left(M - \frac{(a-b)^2}{2}\right)\right]\right)^{-\frac{\Theta^2}{(2M-|a^2-b^2|)^2}}$$

$$\times\left(\frac{2r^2 - 2M + a^2 + b^2 - \zeta\cos(2\theta)}{2r^2 - 2M + a^2 + b^2 + \zeta\cos(2\theta)}\right)^{\frac{2\Sigma\Theta}{(2M-|a^2-b^2|)^2}},$$

$$\Phi(r,\theta) = \frac{\Sigma}{2\,(2M - |a^2 - b^2|)}\ln\left(\frac{2r^2 - 2M + a^2 + b^2 - \zeta}{2r^2 - 2M + a^2 + b^2 + \zeta}\right)$$

$$+ \frac{\Theta}{2\,(2M - |a^2 - b^2|)}\ln\left(\frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}\right), \quad \zeta = \sqrt{(a^2 - b^2)^2 + 4M\,[M - (a^2 + b^2)]}. \tag{27}$$

- In analogy with the four-dimensional case, it is natural to interpret the static limit as a five-dimensional ZV-FJNW spacetime

Final remarks

- Several solutions to the Einstein-Scalar system have been explored in the literature. We relate all these solutions to the general result of Eriș and Gürses. The multipolar modification of the resulting line elements is simply the effect produced by the scalar field
- In five dimensions, we observed that the static limit provides a ZV-FJNW spacetime. However, no higher-dimensional ZV spacetime in vacuum has yet been constructed in the literature
- Conformal and disformal transformations serve to construct rotating solutions in some other theories. Rotating BBMB
- Inclusion of the cosmological constant

Gracias!