Stellar modeling in light of current astrophysical constraints: radial oscillations of relativistic stars made of anisotropic matter

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Concepción, 5-7 November 2025

Talk based on two works:

1) G.P. Published in Universe 11 (2025) 146 (arXiv 2504.20347)

Exact analytic solution without assuming an EoS.

2) C. Sepúlveda, G. P, submitted to Journal of High Energy Astrophysics

Quark stars assuming a certain EoS, numerical analysis.

It is about compact stars (dense, relativistic stars).

Recall yesterday's talk by A. Saavedra on compact stars.

Outline

1. Interior solutions
2. Criteria for realistic solutions / Astrophysical constraints
3. Tidal Love numbers
4. Equation-of-state
5. Radial oscillations
6. Main results
7. Conclusions

Motivation: simple and two-fold

1. Obtain exact analytic solutions to Einstein's field equations (not trivial, always challenging).

2. What is the Equation-of-State of compact objects?

Set up

Einstein's field equations (interior solution r $\leq R$) :

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab} \rightarrow \nabla_a T^{ab} = 0$$

For fluid spheres:

$$T_a^b = Diag[-\rho, p_r, p_t, p_t], \quad \Delta = p_t - p_r$$

For metric tensor:

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2[d\theta^2 + sin^2(\theta) d\phi^2]$$

Interactions of relativistic particles in dense media may generate anisotropies.

Structure Equations

Exterior vacuum solution (r $\geq R$)

$G_{ab} = 0 \rightarrow R_{ab} = 0$

Schwarzschild geometry (1916)

$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2[d\theta^2 + sin^2(\theta)d\phi^2]$

M: stellar mass

Interior solution: Define mass function

$e^{2B(r)} = \frac{1}{1-2\frac{m(r)}{r}}$

TOV equations

m'(r) $= 4\ \pi r^2 \rho(r)$

$\nu'(r) = 2[m(r) + 4\pi r^3 p(r)]\ \frac{1}{r^2}\ \frac{1}{1-2\frac{m(r)}{r}}$

p'(r) $=$ -$[\rho(r) + p_r(r)]\frac{\nu'(r)}{2} + \frac{2\Delta}{r}$

Plus some equation-of-state p$_r = f(\rho)$.

Conditions at center (r $\rightarrow 0) : m(0) = 0, \quad p_r(0) = p_c$

Matching conditions at surface (r $\rightarrow R) : p_r(R) = 0, \quad m(R) = R, \quad e^{2\nu(R)} = 1 - 2\frac{M}{R}$.

Numerical vs Analytic solutions

I) Numerical: Structure equations plus i) EoS and ii) a form of anisotropy, for instance:

$\Delta = \kappa \, \frac{2m(r)}{r} \, p_r(r) \quad (here: \; \kappa = 0.8)$

Ensures that $\Delta(0) = 0 = \Delta(R)$

II) Analytic: Impose the Karmarkar condition

$\mathrm{R}_{1414} = \frac{R_{1212} R_{3434} + R_{1224} R_{1334}}{R_{2323}}$

which implies: $\mathrm{e}^{\nu} = \left(C_1 + \frac{1}{C_2} \int dr \sqrt{e^{\lambda} - 1} \right)^2 .$

Assume a mass function m(r) $\rightarrow \lambda(r) \rightarrow \nu(r) \rightarrow (\rho, p_r, \Delta)$

Criteria for realistic solutions

1. Causality:

   $$0 \leq c_s^2 \leq 1$$

2. Buchdahl limit:

   $$\mathrm{M/R} \leq (4/9)$$

3. Energy conditions:

   $$\rho \geq 0, \qquad \rho + p_r + 2p_t \geq 0, \qquad \rho \pm p_i \geq 0, \quad i = r, t.$$

4. Stability based on the relativistic adiabatic index $\Gamma = c_s^2(1 + \rho/p)$

   $$\langle \Gamma \rangle \geq \Gamma_{cr}$$

   The critical value is given by

   $$\Gamma_{cr} = \frac{4}{3} + \frac{19}{21} \, \frac{M}{R}$$

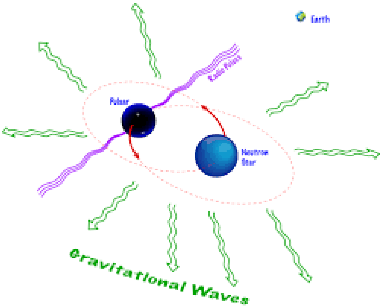
   while the mean value is computed by

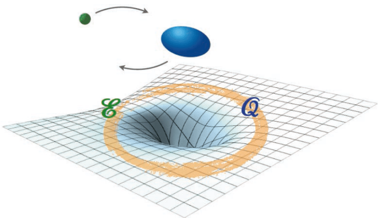
   $$\langle \Gamma \rangle = \frac{\int_0^R dr \, \Gamma(r) p(r) r^2 e^{(\lambda+3\nu)/2}}{\int_0^R dr \, p(r) r^2 e^{(\lambda+3\nu)/2}}.$$

Astrophysical constraints

1. Most massive pulsars (J0348+0432 and J1614-2230) $M \sim 2M_{\odot}$
2. GW events 170817 (NS-NS, $\Lambda_{1.4} = 70 - 580$) and 190417 (BH-companion star $M \approx 2.5\ M_{\odot}$)
3. Light HESS J1731-347 compact object ($M = 0.77\ M_{\odot}$, $R = 10.4\ km$)
4. NICER (Neutron star Interior Composition Explorer) results

Binary systems





Binary system

Classical tests of GR (4)

1. **Gravitational red-shift** (Einstein 1907) (photons lose energy as they propagate outwards)
2. Perihelion advance of planet Mercury (Einstein, 1914)
3. Light deflection (Einstein 1915, Eddington 1919)
4. **Shapiro time delay** (Shapiro 1964) (gravitational field slows down photons)

Binary system

2 stars: Pulsar (primary, mass M_p) and the companion star (mass M_c)

If the pulsar is observed at X-rays, then:

i) Shapiro time delay $\rightarrow$ Mass of the companion star

ii) Keplerian parameters (such as period T and large semi-axis a) $\rightarrow$ Mass of the pulsar

$$\mathrm{a}^3 = \left(\frac{G}{4\pi^2}\right) T^2 \left(M_p + M_c\right)$$

iii) Gravitation red-shift $\rightarrow$ determine the radius of the pulsar

$$\mathrm{Z}_G + 1 = \frac{1}{\sqrt{f(R_p)}}, \quad f(r) = 1 - \frac{2M_p}{r}$$

Tidal deformability

Tidal field (star 1):

$$\mathrm{E}_{ij} = \frac{\partial^2 \Phi_{ext}}{\partial x^i \partial x^j}$$

Cuadrupolar moment (star 2):

$$\mathrm{Q}_{ij} = \int d^3x \delta\rho(\vec{x}) \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij}\right)$$

Deformabilidad:

$$\mathrm{Q}_{ij} = -\lambda\, E_{ij}$$

$$\lambda = \frac{2}{3} k R^5, \qquad \Lambda = \frac{2}{3}\, \frac{k}{C^5}$$

Gravito-electric tidal Love numbers

Equation for the metric perturbations H(r) (multipole l=2)

H''(r) + P(r) H'(r) + Q(r) H(r) = 0, $\quad$ H(r) $\sim r^2, r \to 0$

Define: y(r) = r (H'(r)/H(r))

Satisfies the Riccati equation with y(0) = 2 (=l)

r y(r)' + y(r)$^2 + (rP(r) - 1)y(r) + r^2 Q(r) = 0, \quad y_R = y(R)$

Tidal Love number k_2

In terms of compactness C=M/R and y_R :

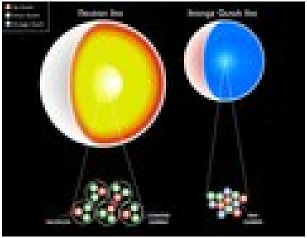
$$k_2 = \frac{8C^5}{5} \frac{K_o}{3\, K_o\, \ln(1-2C)+P_5(C)},$$

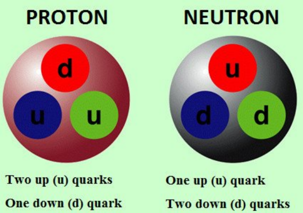
$$K_o = (1-2C)^2\, [2C(y_R-1)-y_R+2]$$

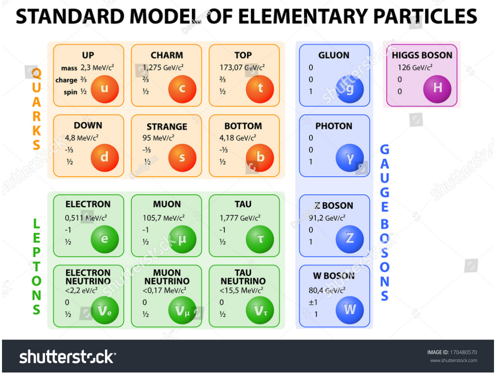
$$P_5(C) = \; 2C\, \Big(4C^4(y_R+1)+2C^3(3y_R-2)+ \\ 2\, C^2(13-11y_R)+3C(5y_R-8)-3y_R+6\Big)$$

Black Holes (Schwarzschild in GR): $C \to 1/2, \quad k_2 \to 0$

Quark stars




PROTON
d
u
u
Two up (u) quarks
One down (d) quark
NEUTRON
u
d
d
One up (u) quark
Two down (d) quarks


STANDARD MODEL OF ELEMENTARY PARTICLES
QUARKS
UP
mass 2,3 MeV/c²
charge ⅔
spin ½
u
CHARM
1,275 GeV/c²
⅔
½
c
TOP
173,07 GeV/c²
⅔
½
t
DOWN
4,8 MeV/c²
-⅓
½
d
STRANGE
95 MeV/c²
-⅓
½
s
BOTTOM
4,18 GeV/c²
-⅓
½
b
LEPTONS
ELECTRON
0,511 MeV/c²
-1
½
e
MUON
105,7 MeV/c²
-1
½
μ
TAU
1,777 GeV/c²
-1
½
τ
ELECTRON NEUTRINO
<2,2 eV/c²
0
½
νe
MUON NEUTRINO
<0,17 MeV/c²
0
½
νμ
TAU NEUTRINO
<15,5 MeV/c²
0
½
ντ
GAUGE BOSONS
GLUON
0
0
1
g
PHOTON
0
0
1
γ
Z BOSON
91,2 GeV/c²
0
1
Z
W BOSON
80,4 GeV/c²
±1
1
W
HIGGS BOSON
126 GeV/c²
0
0
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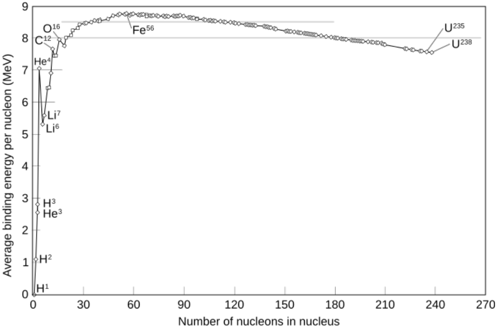
Quark stars

Witten Hypothesis (1984): Quark matter is more stable that Iron 56 (true ground state of QCD)



Quark stars: Analytic EoS

3 parameters: Bag constant $B = 80 MeV/fm^3$, mass of s quark $m_s = 150 MeV$, QCD superconducting energy gap $\Delta = 150 MeV$

$$\alpha = \frac{2\Delta^2}{3} - \frac{m_s^2}{6}$$

$$p(\rho) = \frac{\rho(r)}{3} - \frac{4B}{3} + \frac{3\alpha\mu^2}{\pi^2},$$

$$\mu^2 = \left[\frac{4}{9}\pi^2(\rho(r) - B) + \alpha^2\right]^{1/2} - \alpha.$$

In the limit $\alpha \to 0$, simplified MIT bag model

$$p = (1/3)\,(\rho - 4B)$$

Radial oscillations of pulsating stars

Define:

$$\xi \equiv \frac{\Delta r}{r}, \qquad \eta \equiv \frac{\Delta P}{P}$$

Equations for perturbations (system of 2 coupled ODEs):

$$\xi'(r) = -\frac{1}{r}\left(3\xi + \frac{\eta}{\Gamma}\right) - \frac{P'(r)}{P+\rho}\xi(r)$$

$$\eta'(r) = \xi\left[\omega^2 r(1+\rho/P)e^{\lambda-\nu} - \frac{4P'(r)}{P} - 8\pi(P+\rho)re^{\lambda} + \frac{r(P'(r))^2}{P(P+\rho)}\right] + \eta\left[-\frac{\rho P'(r)}{P(P+\rho)} - 4\pi(P+\rho)re^{\lambda}\right]$$

Boundary value problem

Throughout the star from center to surface

$0 \leq r \leq R$

i) Condition at the center (r $\rightarrow$ 0)

$\eta(0) = -3\Gamma(0)\xi(0)$

ii) Condition at the surface (r $\rightarrow R$)

$$\eta(R) = \xi(R)\left[-4 + (1 - 2M/R)^{-1}\left(-\frac{M}{R} - \frac{\omega^2 R^3}{M}\right)\right]$$

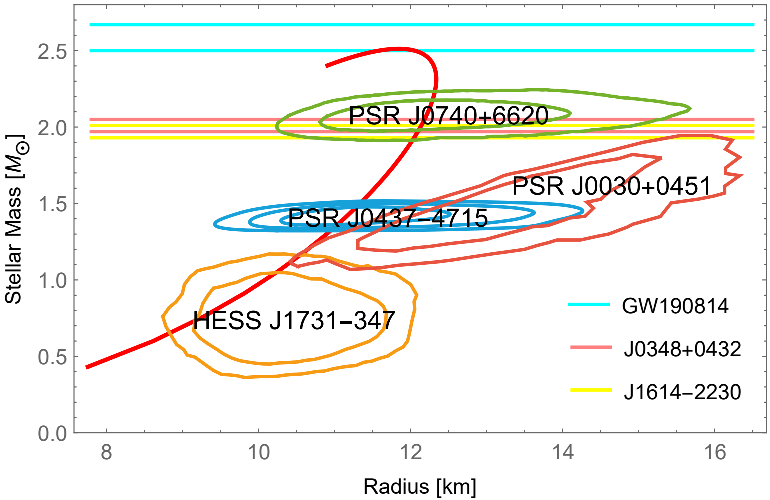
Discrete frequency spectrum ω_n $(n = 0, 1, 2, ...)$

$\omega_0^2 < \omega_1^2 < ... < \omega_n^2$

Seismic parameter: Large frequency separation $\Delta\nu_n = \nu_{n+1} - \nu_n$

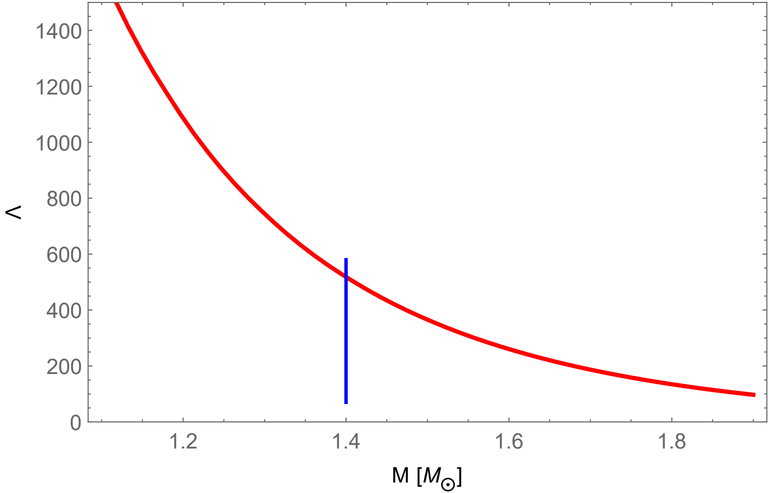
Results work with C. Sepúlveda

Main numerical results I



Mass-to-radius relationship and current astrophysical constraints

Main numerical results II



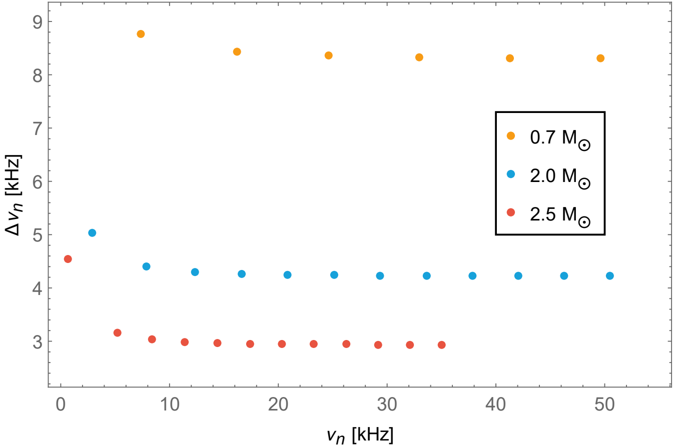
Dimensionless deformability versus stellar mass and limits from the GW event 170817.

Spectra for 3 stellar masses

Mode order n	0.7 $M_\odot$	2.0 $M_\odot$	2.5 $M_\odot$
0	7.40	2.86	0.69
1	16.16	7.90	5.23
2	24.60	12.30	8.39
3	32.95	16.61	11.43
4	41.29	20.88	14.41
5	49.60	25.13	17.38
6	57.91	29.37	20.34
7	66.21	33.61	23.29
8	74.52	37.84	26.23
9	82.81	42.06	29.17
10	91.11	46.29	32.11
11	99.41	50.52	35.05
12	107.70	54.74	37.99

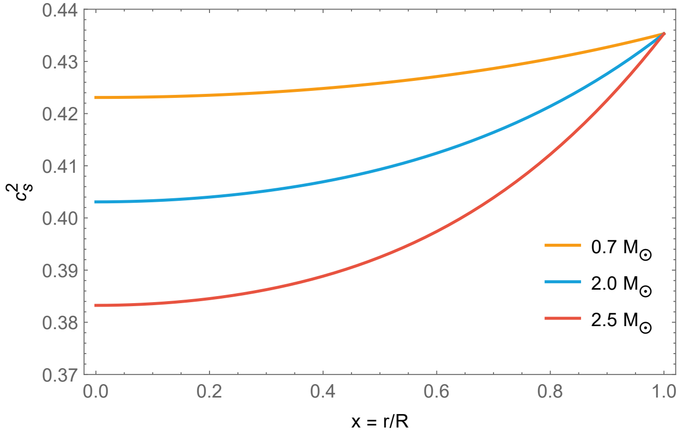
Table: Frequencies (in kHz) of the radial oscillation for the 13 lowest modes considering three different stellar masses: 0.7 $M_\odot$, 2.0 $M_\odot$, and 2.5 $M_\odot$.

Large frequency separations



Large frequency separations $\Delta\nu_n$ in kHz, for 0.7 $M_\odot$ in orange, 2.0 $M_\odot$ in blue, and 2.5 $M_\odot$ in red. We consider the first 13 oscillation modes. Due to the high values of the frequencies for 0.7 $M_\odot$, only the first 6 frequencies are displayed, instead of 13. For all values, see Table 2.

Sound speeds



Radial speed of sound, c_s^2, versus normalized radial coordinate, r/R, for 0.7 $M_\odot$ in yellow, 2.0 $M_\odot$ in blue, and 2.5 $M_\odot$ in red.

Results work I published alone

Anisotropic star via Karmarkar condition

Assume a metric potential:

$$\nu(r) = C + \left(\frac{r}{A}\right)^2$$

Karmarkar condition fixes the second metric potential

$$e^{\lambda} = 1 + \frac{B^2 r^2}{A^4} exp(C + (r/A)^2)$$

Quantities are now computed one by one using TOV Eqns

$$\mathrm{m(r)} = \frac{B^2 r^3 exp(C + (r/A)^2)}{2[A^4+B^2r^2exp(C+(r/A)^2)]}$$

$$\rho(r) = \frac{B^2 exp(C+(r/A)^2)[3A^4+2A^2r^2+B^2r^2exp(C+(r/A)^2)]}{8\pi[A^4+B^2r^2exp(C+(r/A)^2)]^2}, \qquad \rho_c = \rho(0) = \frac{3B^2e^C}{8\pi A^4}$$

$$\mathrm{p}_r(r) = \frac{2A^2-B^2exp(C+(r/A)^2)}{8\pi[A^4+B^2r^2exp(C+(r/A)^2)]}, \qquad p_c = p_r(0) = \frac{2-(B/A)^2e^C}{8\pi A^2}$$

$$\Delta(r) = \frac{[A^2-B^2exp(C+(r/A)^2)]^2r^2}{8\pi[A^4+B^2r^2exp(C+(r/A)^2)]^2}, \qquad \Delta(0) = 0$$

There is a radius:

$$\mathrm{R}=\mathrm{A}\sqrt{-C + ln\left(\frac{2A^2}{B^2}\right)}$$

Isotropic star: Tolman solution

Metric tensor:

$$e^{\lambda(r)} = \frac{1+2(r/a)^2}{[1-(r/r_0)^2][1+(r/a)^2]}$$

$$e^{\nu(r)} = b^2[1 + (r/a)^2]$$

Matter content:

$$\mathrm{p(r)} = 1/(8\ \pi r_0^2)\ \frac{r_0^2-a^2-3r^2}{2r^2+a^2}$$

$$\mathrm{m(r)} = \mathrm{r}^3/(2r_0^2)\ \frac{r^2+a^2+r_0^2}{a^2+2r^2}$$

$$\rho(r) = \frac{1}{8\pi r_0^2}\ \frac{6r^4+(7a^2+2r_0^2)r^2+3a^4+3a^2r_0^2}{(a^2+2r^2)^2}$$

There is a radius:

$$\mathrm{R} = \sqrt{r_0^2 - a^2}\,(1/\sqrt{3})$$

Modeling HESS compact object

M=0.8 $M_\odot$, $\quad R = 10.4\ km$

1. Case Anisotropic star (Karmarkar condition):
$A = 27.90\ km$, $B = 44.64\ km$, $C = -0.39$.

2. Case Isotropic star (Tolman solution):
$a = 25.07\ km$, $r_0 = 30.89\ km$, $b = 0.81$.

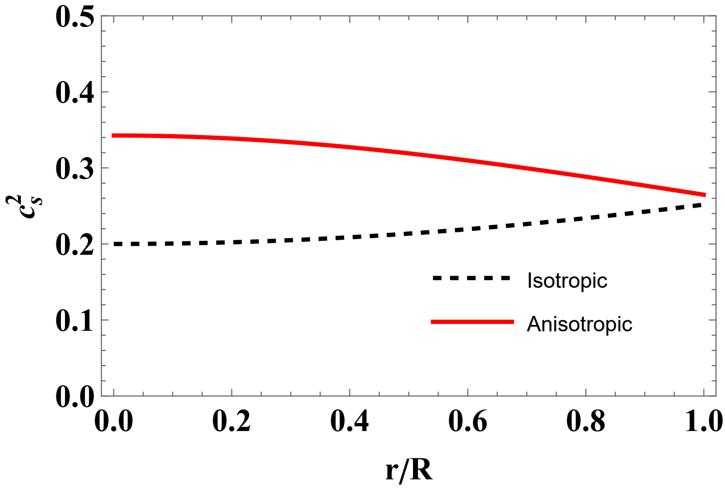
All well-established criteria are met.

Spectra comparison

Table: Frequencies (in kHz) of radial oscillation modes both for isotropic and anisotropic stars. We have considered $M = 0.8M_{\odot}$ and $R = 10.42km$ in the case of the isotropic object, and $M = 0.77M_{\odot}$ and $R = 10.47km$ in the case of the anisotropic star.

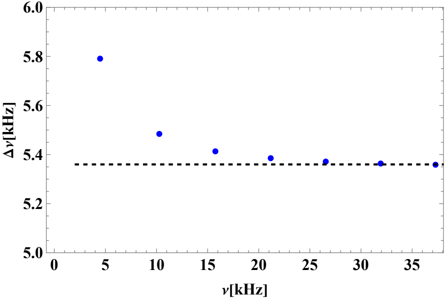
Mode order n	Isotropic star	Anisotropic star
0	4.48	5.81
1	10.27	12.62
2	15.76	19.20
3	21.17	25.72
4	26.55	32.22
5	31.93	38.72
6	37.29	45.20
7	42.65	51.68

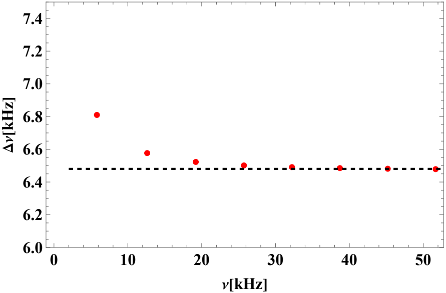
Sound speed



Speed of sound versus radial coordinate for anisotropic star (red solid curve) and isotropic object (black dashed curve).

Large frequency separations





Large frequency separations (Tolman: upper panel, Karmarkar lower panel).

Conclusions

1. Studied anisotropic quark stars numerically (assuming certain EoS and anisotropy)
2. Computed M-R profile and tidal deformability: Satisfy current astrophysical constraints
3. Computed spectra and large frequency separations assuming 3 different stellar masses (0.7, 2.0 and 2.5 solar masses)
4. Obtained exact analytic solution via Karmarkar condition
5. Modeled HESS object $\rightarrow$ numerical values of the 3 free parameters
6. Studied radial oscillations: computed frequencies and large frequency separations
7. Comparison to the Tolman IV solution (isotropic stars)

Thank you for your attention!