On the wave zone in $f(R)$ gravity

Encuentro CosmoConce y partículas

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Outline

1. Linear theory versus asymptotic expansion in GR
2. $f(R)$ gravity theory
3. The wave zone in $f(R)$ gravity
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Linearized Regime [1, 2]

- Background decomposition:

$$g_{ij} = \delta_{ij} + f_{ij}, \qquad |f_{ij}| \ll 1.$$

- Linearized ADM constraints:

$$\mathcal{H}: \ \partial_i \partial_j f^{ij} - \Delta f = 0, \qquad \mathcal{H}_i: \ \partial_j p^{ij} = 0.$$

- These are **elliptic equations** on each spatial slice. They determine the trace and longitudinal parts of f_{ij}, leaving only the **transverse–traceless (TT)** sector unconstrained

$$f_{ij} = f_{ij}^{TT} + f_{ij}^{T} + f_{ij}^{L}.$$

- The constraints therefore define the non-radiative sector of the field.

Linearized Theory vs. Wave Zone in General Relativity

Wave-Zone Regime [1, 2]

- Evolution equations yield the hyperbolic system:

$$\ddot{f}_{ij}^{TT} - \Delta f_{ij}^{TT} = 0.$$

- Outgoing solution:

$$f_{ij}^{TT}(t, r, \Omega) = \frac{F_{ij}(t - r, \Omega)}{r} + O(r^{-2}).$$

- Asymptotic structure:

$$\text{static} \sim 1/r^2, \qquad \text{oscillatory} \sim F_{ij}(t - r)/r.$$

- $f^T \lesssim B(\Omega)/r + A(t, \Omega)e^{ikr}/r^2$, static part (Newtonian background).

Equations of motion

$$
\begin{aligned}
\dot{g}_{ij} &\equiv \{g_{ij}, H\} = \frac{2N}{\sqrt{g}}\left(\pi_{ij} - \frac{1}{2}g_{ij}\pi\right) + \nabla_{(j}N_{i)}, \\
\dot{\pi}^{ij} &\equiv \left\{\pi^{ij}, H\right\} = -N\sqrt{g}\left(R_{ij} - \frac{1}{2}g^{ij}R\right) + \frac{1}{2}\frac{N}{\sqrt{g}}g^{ij}\left(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2\right) \\
&-\frac{2N}{\sqrt{g}}\left(\pi^{im}\pi_m{}^j - \frac{1}{2}\pi\pi^{ij}\right) + \sqrt{g}\left(\nabla^i\nabla^j N - g^{ij}\nabla_m\nabla^m N\right) \\
&+\nabla_m\left(\pi^{ij}N^m\right) - \pi^{mj}\nabla_m N^i - \pi^{mi}\nabla_m N^j.
\end{aligned}
\tag{1}
$$

Wave zone hypothesis

The wave zone is well-defined if and only if the following considerations are taking into account [2]:

1. $kr \gg 1$, finite wave energy.
2. asymptotic behavior,

$$\left\{ |g_{ij} - \delta_{ij}| \,, \quad |N - 1|, \quad \left|N^i\right| \right\} \lesssim A/r \ll 1. \tag{2}$$

3. Controlling non-linear contributions,

$$\begin{gathered} |\partial g_{ij}/\partial(kr)|^2 \ll |g_{ij} - \delta_{ij}| \,, \\ |\partial N_i/\partial(kr)|^2 \ll |N_i| \,, \quad |\partial N/\partial(kr)|^2 \ll |N - 1|. \end{gathered} \tag{3}$$

In the Jordan frame, $f(R)$ gravity theory is expressed as follows [3]

$$S\left[g_{\mu\nu}\right] = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}f(R), \tag{4}$$

or equivalently by [4]

$$S\left[g_{\mu\nu}, s, \phi\right] = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left[f(s) - \phi(s - R)\right]. \tag{5}$$

Gravitational waves in $f(R)$ gravity

The linearized approach shows that this theory propagates three degrees of freedom: The TT modes for the gravitational sector and the so-called scalaron. Therefore, the gravitational waves in this theory propagates at [5]

$$v_{\mathrm{GW}}^2 = c^2 \left(1 + \frac{\Delta f(R)}{R}\right). \tag{6}$$

However, from recent observational data we found [6]

$$-3 \times 10^{-15} \leq \frac{v_{\mathrm{GW}} - c}{c} \leq 7 \times 10^{-16}. \tag{7}$$

With this constraint a lot of modified gravity theories have been ruled out.

The hamiltonian for $f(R)$ gravity is given by

$$\pi^{ij} = \frac{\sqrt{g}}{2}\left[\phi\left({}_{\mathbb{T}}K^{ij} - \frac{2}{3}Kg^{ij}\right) - \frac{g^{ij}}{N}\left(\dot{\phi} - N^i\partial_i\phi\right)\right], \quad p = -\sqrt{g}K, \tag{8}$$

$$\dot{g}_{ij} = \frac{N}{\sqrt{g}}\left(\frac{4_{\mathbb{T}}\pi_{ij}}{\phi} - \frac{2}{3}pg_{ij}\right) + \nabla_i N_j + \nabla_j N_i, \quad \dot{\phi} = \frac{2N}{3\sqrt{g}}(\phi p - \pi) + N^i\partial_i\phi, \tag{9}$$

thus

$$H \equiv H_{\mathrm{J}}^{*} + \partial_i\left(2\pi^{ij}N_j - \sqrt{g}\phi\partial^i N\right), \quad \text{where} \quad H_{\mathrm{J}}^{*} = \sqrt{h}\left(NC + N^i C_i\right), \tag{10}$$

being the constraints

$$C = \frac{2}{g}\left(\frac{\mathbb{T}\pi \cdot \mathbb{T}\pi}{\phi} + \frac{1}{6}\phi p^2 - \frac{1}{3}p\pi\right) + \frac{1}{2}\left(\phi s - f(s) - \phi R + 2\nabla_i \nabla^i \phi\right),$$

$$C_i = -2\nabla_j\left(\frac{\pi^j_{\ i}}{\sqrt{g}}\right) + \frac{\pi}{\sqrt{g}}\partial_i \phi, \tag{11}$$

The gravitational modes satisfy

$$g_{ij} - \delta_{ij} \lesssim \frac{B_{ij}(\Omega)}{r} + \frac{A_{ij}(\Omega)e^{ikr}}{r}, \tag{12}$$

$$\pi^{lj} \lesssim \frac{B^{lj}(\Omega)}{r^2} + k\frac{A^{lj}(\Omega)e^{ikr}}{r}. \tag{13}$$

The wave zone in $f(R)$ gravity

Solving the constraints, we can estimate the scalar field asymptotic behavior. This leads to

$$\frac{p}{\sqrt{g}} = -K = \mathcal{O}\left(r^{-2}\right) \tag{14}$$

Next, from the constraint $C_i = 0$ one obtains

$$\partial_i \phi = \frac{2\nabla_j \left(\pi^j{}_i / \sqrt{g}\right)}{p/\sqrt{g}} \sim \frac{\mathcal{O}\left(r^{-3}\right)}{\mathcal{O}\left(r^{-2}\right)} = \mathcal{O}\left(r^{-1}\right). \tag{15}$$

That is,

$$\partial_i \phi \sim \frac{1}{r} \quad \Longrightarrow \quad \phi(r,\Omega) = \phi_0 + \frac{A(\Omega)}{r} + \mathcal{O}\left(r^{-2}\right). \tag{16}$$

Then, the scalar behaves as

$$\phi - \phi_0 \lesssim \frac{Q(\Omega) e^{-m_s r}}{r} + \frac{A(\Omega) e^{i(kr - \omega t)}}{r}, \qquad \left(k = \sqrt{\omega^2 - m_s^2}\right). \tag{17}$$

Asymptotic decomposition

$$\phi(t, r, \Omega) = \underbrace{\phi_0}_{\text{background}} + \underbrace{\frac{Q(\Omega)}{r}}_{\text{newtonian}} + \underbrace{\frac{A(\Omega)\, e^{i(kr - \omega t)}}{r}}_{\text{radiative}} + \mathcal{O}\left(\frac{1}{r^2}\right),$$

$$m_s^2 = \frac{f'(R_0) - R_0 f''(R_0)}{3 f''(R_0)}.$$

Restrictions on the $f(R)$ gravity wave zone

Condition	**Interpretation**	**Comment**
$f'(R_0) > 0$	Positive effective coupling	Prevent ghost fields
$f''(R_0) > 0$	Stability (real mass)	$m_s^2 > 0$ Ensure regular propagation.
$0 < f''(R_0) < \infty$	Regular propagation	Compatible with $\partial_i \phi \sim 1/r$.

$f(R)$ models: Admissible vs. Non admissible

Models	$f'(R)$	$f''(R)$
Admissible		
$R + \alpha R^2$ [5] $(\alpha > 0)$	> 0	> 0
$f(R) = R - m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}$ [5]	> 0	> 0
Non admissible		
$R - \mu^4/R$ [7]	> 0	< 0
$R - 2\Lambda$ (GR+Λ)	> 0	$= 0$

Total ADM–type energy (Jordan frame):

$$E_{\text{tot}}^{f(R)} = \frac{1}{16\pi} \oint_{S_\infty} dS_i \left[\phi_0(\partial_j f_{ij} - \partial_i f) \; - \; 2\,\partial_i(\delta\phi) \right].$$

Decomposition:

$$E_{\text{tot}}^{f(R)} = E_{\text{grav}} + E_{\text{scalar}},$$

$$\boxed{\begin{aligned} E_{\text{grav}} &= \frac{\phi_0}{16\pi} \oint_{S_\infty} dS_i\,(\partial_j f_{ij} - \partial_i f), \\ E_{\text{scalar}} &= -\frac{1}{8\pi} \oint_{S_\infty} dS_i\,\partial_i(\delta\phi) = \frac{1}{8\pi} \int Q(\Omega)\,d\Omega. \end{aligned}}$$

Asymptotic behavior:

$$f_{ij} = \delta_{ij} + \frac{f_{ij}(\Omega)}{r} + O(r^{-2}), \qquad \delta\phi = \frac{Q(\Omega)}{r} + O(r^{-2}).$$

Gravitational and scalar energy contributions

Interpretation:

- $\phi_0 = f'(R_0)$ rescales the effective Newton constant $G_{\rm eff} = G/\phi_0$.
- $E_{\rm grav}$ is the usual ADM energy multiplied by ϕ_0.
- $E_{\rm scalar}$ comes only from the *static tail* of the scalar field: Newtonian tail ($R_0 = 0$): $\delta\phi \sim A/r \Rightarrow$ possible monopole contribution.
- Radiative parts ($\sim e^{i\omega(t-r)}/r$) do not modify $E_{\rm tot}$ at leading order; they appear only in the energy flux.

Summary: The total conserved energy in $f(R)$ gravity is determined by the gravitational $1/r$ field and the static (Newtonian) tail of the scalar, evaluated on the asymptotic 2–sphere at infinity.

Conclusions

1. In the ADM formulation of $f(R)$, the momentum constraint fixes $\partial_i \phi \sim 1/r$ in the wave zone region, with $\phi = f'(R)$.
2. The decomposition of the scalar field shows two pieces:
(i) The term ($\sim Q(\Omega)/r$) that corrects the Newtonian background, and (ii) a radiative term ($\sim A(\Omega)e^{i(kr-\omega t)}/r$).
3. The viability conditions coincide with those derived from stability: $f'(R) > 0$ and $f''(R) > 0$, guaranteeing regular propagation and avoiding Dolgov-Kawasaki type instabilities.
4. Viable models such as Starobinsky ($R + \alpha R^2$) and Hu-Sawicki meet the conditions and allow scalar radiation; non-viable models such as $f(R) = R - \mu^4/R$ destroy the asymptotic behavior and do not admit a consistent wave region.

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Thank you very much!

Theorem

Be $\varphi \in C^2(\Omega)$ a Poisson's equation solution $\Delta\varphi = -4\pi\rho$ in $\Omega \equiv \mathbb{R}^3/\{0\}$ such that $\varphi \to 0$ if $r \to \infty$. Thus, the solution $\varphi(r,\theta,\phi)$ can be expanded in spherical harmonic $Y_{lm}(\theta,\phi)$,

$$\begin{aligned} \varphi(r,\theta,\phi) &= \sum_{l=0}^{l=\infty}\sum_{m=-l}^{m=l} \chi_{lm}(r) r^l Y_{lm}(\theta,\phi) \\ \chi_{lm}(r) &\equiv \int_r^\infty \frac{M_{lm}(r')}{r^{2l+2}} dr' \\ M_{lm}(r) &\equiv \int_{B_r(0)} \rho(r',\theta',\phi')\, r'^l\, Y'_{lm}(r',\theta',\phi')\, d^3r' \end{aligned} \tag{18}$$

Appendix

1. If in the region $r > R$ the source behaves as $\rho \sim Y_{lm}e^{ikr}/r^n$ then,

$$\varphi \sim \left(1/k^2\right) Y_{lm}\frac{e^{ikr}}{r^n}[1 + \mathcal{O}(1/kr)] + \sum_{l=0}^{l=\infty}\sum_{m=-l}^{m=l} c_{lm} Y_{lm}\frac{1}{r^{l+1}} \tag{19}$$

   where the coefficients c_{lm} come from the interior region $r \leq R$.

2. If the region $r > R$,

$$\psi \sim B_m(t,\theta,\phi)/r^m + A_n(t,\theta,\phi)e^{ikr}/r^n \tag{20}$$

where $|B_m(t,\theta,\phi)|$ y $|A_n(t,\theta,\phi)|$ are bounded functions, then

Appendix

$$
\begin{aligned}
\frac{1}{\Delta}\partial_j\psi &\sim \frac{\langle B\rangle}{r^{m-1}} + \frac{1}{ik}\frac{\langle A\rangle e^{ikr}}{r^n} + \sum_{l=1}^{\infty}\sum_{p=-l}^{p=l} c_{lp}Y_{lp}\frac{1}{r^{l+1}}, \quad (m\geq 2) \\
\frac{1}{\Delta}\partial_i\partial_j\psi &\sim \frac{\langle B\rangle}{r^{m}} + \frac{\langle A\rangle e^{ikr}}{r^n} + \sum_{l=2}^{\infty}\sum_{p=-l}^{p=l} c_{lp}Y_{lp}\frac{1}{r^{l+1}}, \quad (m\geq 1)
\end{aligned}
\tag{21}
$$

where $\langle A\rangle$, represents the angular part average.