Maxwell Chern-Simons gravity in 3D: Thermodynamics of cosmological solutions and black holes with torsion

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Introduction

- Three-dimensional gravity provides an ideal setting to explore conceptual and structural aspects of gravitational theories.
- Unlike in four dimensions, 3D gravity has no local degrees of freedom, which simplifies the dynamics and allows the explicit construction of exact solutions and a detailed analysis of their properties.
- When formulated as a **Chern–Simons (CS)** gauge theory, it naturally links **symmetries**, **topology**, and **geometry** in a unified framework.

Introduction

- Extending the gauge symmetries of General Relativity has proven to be a fruitful strategy to construct new gravity models.
- In particular, **extensions, expansions, and deformations** of the Poincaré and AdS symmetries have led to various CS (super)gravity theories.
- Among them, the **Maxwell algebra**—an extension of the Poincaré algebra—has attracted special attention, as it introduces a new connection: the gravitational Maxwell field.

$$[P_a, P_b] = \epsilon_{abc} Z^c$$

- In particular, the gravitational Maxwell gauge field modifies not only the vacuum energy and angular momentum of the stationary configuration but also the **asymptotic structure**.

Introduction

- A deep understanding of a physical theory requires having control of its asymptotic structure, described by the asymptotic behavior of the physical fields, far away from any physical process.

- The case of three-dimensional gravity with negative cosmological constant has been extensively studied due to the existence of the BTZ black hole and its extraordinarily asymptotic structure, given by two copies of the Virasoro algebra.[J. Brown, M. Henneaux (1986)]

- In the asymptotically flat case, an infinite dimensional asymptotic symmetry algebra can be found at null infinity, given by the $\mathfrak{bms}_3$ algebra.

$$\mathfrak{vir} \oplus \mathfrak{vir} \xrightarrow{\Lambda \to 0} \mathfrak{bms}_3$$

- **The study of the behavior of dynamic fields in the asymptotic region become crucial for computing the thermodynamic properties of configurations with a sensible thermodynamics**.

Introduction

Despite its theoretical appeal, the thermodynamic properties of Maxwell gravity solutions have remained largely unexplored.

In this work, we study the **entropy** and **thermodynamic consistency** of asymptotically flat cosmological solutions in three-dimensional Maxwell gravity.

We extend the analysis to the **torsional case**, obtaining a new black hole solution that generalizes the **BTZ-like black hole with torsion**.

Outline

Maxwell CS gravity

The Maxwell algebra generated by $\{J_a, P_a, Z_a\}$ is given by the following non-vanishing commutators:

$$
\begin{aligned}
[J_a, J_b] &= \epsilon_{abc} J^c\,, \qquad & [J_a, P_b] = \epsilon_{abc} P^c\,, \\
[P_a, P_b] &= \epsilon_{abc} Z^c\,, \qquad & [J_a, Z_b] = \epsilon_{abc} Z^c\,.
\end{aligned}
$$

The gauge connection one-form A for the Maxwell algebra reads

$$A = e^a P_a + \omega^a J_a + \sigma^a Z_a$$

The non-degenerate bilinear form is

$$
\begin{aligned}
\langle J_a J_b \rangle &= \alpha_0 \eta_{ab} \qquad & \langle P_a P_b \rangle = \alpha_2 \eta_{ab} \\
\langle J_a P_b \rangle &= \alpha_1 \eta_{ab} \qquad & \langle J_a Z_b \rangle = \alpha_2 \eta_{ab}
\end{aligned}
$$

where α_0, α_1 and α_2 are arbitrary constants.

Maxwell CS gravity

Then, considering the previous invariant tensor and the one-form gauge connection in the CS action

$$I[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \left\langle AdA + \frac{2}{3}A^3 \right\rangle ,$$

we obtain

$$I_{\mathfrak{max}} = \frac{k}{4\pi} \int \alpha_0 L(\omega) + 2\alpha_1 R_a e^a + \alpha_2 \left(e^a T_a + 2\sigma^a R_a \right) .$$

When $\alpha_2 \neq 0$, the field equations can be written as

$$R^a = 0 , \qquad T^a = 0 , \qquad D\sigma^a + \frac{1}{2}\epsilon^{abc} e_b e_c = 0 .$$

Similarly to GR, the geometries described by these equations of motion are Riemannian (**torsionless**) and locally flat.

Boundary conditions

We propose the following behavior of the gauge fields at the boundary

$$A = g^{-1} dg + g^{-1} a g \,,$$

where the radial dependence is entirely captured by the group element $g = g(r)$. The auxiliary gauge field has the form $a = a_\phi d\phi + a_t dt$, with the angular component given by [P.Concha, N.Merino, O.Miskovic, P.Salgado-Rebolledo, E.Rodríguez, O.Valdivia (2018)]

$$a_\phi = J_1 + \frac{1}{2}\mathcal{M}\, J_0 + \frac{1}{2}\mathcal{N}\, P_0 + \frac{1}{2}\mathcal{F}\, Z_0 \,,$$

where $\mathcal{N}, \mathcal{M}$, and $\mathcal{F}$ stand for arbitrary functions of the boundary coordinates.

Boundary conditions

The asymptotic form of a_ϕ is preserved under a restricted set of gauge transformations, $\delta a = d\lambda + [a, \lambda]$, with the Lie-algebra-valued parameter $\lambda_{(0)} = \lambda_{(0)}[y, f, h]$ generated by

$$\lambda_{(0)}[y, f, h] = yJ_1 - y' J_2 + fP_1 - f' P_2 + hZ_1 - h' Z_2 + \left(\frac{\mathcal{M}}{2} f + \frac{\mathcal{N}}{2} y - f'' \right) P_0 + \left(\frac{\mathcal{M}}{2} y - y'' \right) J_0 + \left(\frac{1}{2} \mathcal{M} h + \frac{1}{2} \mathcal{F} y + \frac{1}{2} \mathcal{N} f - h'' \right) Z_0 \, ,$$

provided that the functions $\mathcal{M}, \mathcal{N}$ and $\mathcal{F}$ transform according to:

$$\begin{aligned} \delta_{(0)} \mathcal{M} &= \mathcal{M}' y + 2\mathcal{M} y' - 2y''' \, , \\ \delta_{(0)} \mathcal{N} &= \mathcal{M}' f + 2\mathcal{M} f' - 2f''' + \mathcal{N}' y + 2\mathcal{N} y' \, , \\ \delta_{(0)} \mathcal{F} &= \mathcal{M}' h + 2\mathcal{M} h' - 2h''' + \mathcal{N}' f + 2\mathcal{N} f' + \mathcal{F}' y + 2\mathcal{F} y' \, . \end{aligned}$$

The asymptotic symmetries are preserved under time evolution by choosing

$$a_t = \lambda_{(0)}[\mu, \xi, \nu] \, ,$$

where μ, ξ, and ν stand for arbitrary functions of the boundary coordinates which are assumed to be fixed at the boundary ("**chemical potentials**")

Asymptotic conditions with chemical potentials

The asymptotic symmetry generators can be computed in the Regge-Teitelboim approach such that their variations reduce to

$$\delta Q[\lambda] = -\frac{k}{2\pi}\int_{\partial\Sigma} \langle \lambda \delta a \rangle \, d\phi \quad \Longrightarrow Q[y,f,h] = -\int d\phi \, (y\mathcal{J} + f\mathcal{P} + h\mathcal{Z}) \ ,$$

$$\mathcal{J} = \frac{k}{4\pi}\left(\alpha_2\mathcal{F} + \alpha_1\mathcal{N} + \alpha_0\mathcal{M}\right) \quad ; \quad \mathcal{P} = \frac{k}{4\pi}\left(\alpha_2\mathcal{N} + \alpha_1\mathcal{M}\right) \quad ; \quad \mathcal{Z} = \frac{k}{4\pi}\alpha_2\mathcal{M} \, .$$

The transformations law of the dynamical variables $\{\mathcal{J}, \mathcal{P}, \mathcal{Z}\}$ explicitly read

$$\delta\mathcal{J} = 2\mathcal{J}y' + \mathcal{J}'y + 2\mathcal{P}f' + \mathcal{P}'f + 2\mathcal{Z}h' + \mathcal{Z}'h - \frac{\alpha_0 k}{2\pi}y''' - \frac{\alpha_1 k}{2\pi}f''' - \frac{\alpha_2 k}{2\pi}h''' \, ,$$

$$\delta\mathcal{P} = 2\mathcal{P}y' + \mathcal{P}'y + 2\mathcal{Z}f' + \mathcal{Z}'f - \frac{\alpha_1 k}{2\pi}y''' - \frac{\alpha_2 k}{2\pi}f''' \, ,$$

$$\delta\mathcal{Z} = 2\mathcal{Z}y' + \mathcal{Z}'y - \frac{\alpha_2 k}{2\pi}y''' \, .$$

Asymptotic symmetry algebra

The algebra of the conserved charges Q can be readily obtained from the transformation law of the dynamical fields by virtue of $\delta_{\lambda_2} Q[\lambda_1] = \{Q[\lambda_1], Q[\lambda_2]\}$. Thus, expanding in Fourier modes according to $X = \frac{1}{2\pi}\sum X_m e^{im\phi}$, the nontrivial Poisson brackets are given by [P.Concha, N.Merino, O.Miskovic, P.Salgado-Rebolledo, E.Rodríguez, O.Valdivia (2018)]

max-bms$_3$

$$
\begin{aligned}
i\{\mathcal{J}_m, \mathcal{J}_n\} &= (m-n)\,\mathcal{J}_{m+n} + c_{\mathcal{J}}\, m(m^2-1)\delta_{m+n,0}\,, \\
i\{\mathcal{J}_m, \mathcal{P}_n\} &= (m-n)\,\mathcal{P}_{m+n} + c_{\mathcal{P}}\, m(m^2-1)\delta_{m+n,0}\,, \\
i\{\mathcal{J}_m, \mathcal{Z}_n\} &= (m-n)\,\mathcal{Z}_{m+n} + c_{\mathcal{Z}}\, m(m^2-1)\delta_{m+n,0}\,, \\
i\{\mathcal{P}_m, \mathcal{P}_n\} &= (m-n)\,\mathcal{Z}_{m+n} + c_{\mathcal{Z}}\, m(m^2-1)\delta_{m+n,0}\,,
\end{aligned}
$$

where the central extensions $c_{\mathcal{J}}$, $c_{\mathcal{P}}$ and $c_{\mathcal{Z}}$ are fully determined in terms of the constants of the CS action according to

$$
c_{\mathcal{J}} = k\alpha_0 \qquad c_{\mathcal{P}} = k\alpha_1\,, \qquad c_{\mathcal{Z}} = k\alpha_2\,.
$$

Thermodynamics of asymptotically flat cosmological solutions

The entropy can be suitably computed by the general formula

$$S = \frac{k}{2\pi}\left[\int d\tau d\phi \langle A_\tau A_\phi\rangle\right]_{\text{on-shell}} = k\left[\langle a_\tau a_\phi\rangle\right]_{\text{on-shell}} ,$$

For the present field configuration the entropy reads

$$S = k\left[\alpha_0 \mu \mathcal{M} + \alpha_1\left(\mu\mathcal{N} + \xi\mathcal{M}\right) + \alpha_2\left(\nu\mathcal{M} + \xi\mathcal{N} + \mu\mathcal{F}\right)\right] ,$$

where it is assumed that the chemical potentials (ξ, μ, ν) are constrained to fulfill the **regularity conditions**:

- Find a permissible gauge transformations such that $e_t = \sigma_t = 0$.

$$g = e^{\lambda P_2 + \rho Z_2}$$

- Require that the holonomy along the thermal circle to be trivial,

$$H = e^{a_\tau}\Big|_{\text{on-shell}} = (-1)^{n_{\mathfrak{m}}} 1_{2\times 2}$$

Thermodynamics of asymptotically flat cosmological solutions

In summary, the regularity conditions of the Euclidean gauge fields lead to the following chemical potentials

$$\xi = -\frac{n_{\mathfrak{m}} \pi \mathcal{N}}{\mathcal{M}^{3/2}}\,, \qquad \mu = \frac{2\pi n_{\mathfrak{m}}}{\mathcal{M}^{1/2}}\,, \qquad \nu = -n_{\mathfrak{m}} \pi \left[\frac{\mathcal{F}}{\mathcal{M}^{3/2}} - \frac{3\mathcal{N}^2}{4\mathcal{M}^{5/2}}\right].$$

Then, in terms of the variables $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{F}$, the entropy reads

$$S = 2\pi n_{\mathfrak{m}} k \left[\alpha_0 \sqrt{\mathcal{M}} + \alpha_1 \frac{\mathcal{N}}{2\sqrt{\mathcal{M}}} + \frac{\alpha_2}{2}\left(\frac{\mathcal{F}}{\sqrt{\mathcal{M}}} - \frac{\mathcal{N}^2}{4\mathcal{M}^{3/2}}\right)\right].$$

In terms of the (extensive) global charges $\mathcal{J}, \mathcal{P}$ and $\mathcal{Z}$, the entropy can be rewritten as follows

$$S_C = 2\pi n_{\mathfrak{m}} \sqrt{\pi k} \sqrt{\alpha_2} \left[\frac{\mathcal{J}}{\sqrt{\mathcal{Z}}} + \frac{\alpha_0}{\alpha_2}\sqrt{\mathcal{Z}} - \frac{1}{4}\left(\frac{\mathcal{P}}{\mathcal{Z}} - \frac{\alpha_1}{\alpha_2}\right)^2 \sqrt{\mathcal{Z}}\right].$$

Thermodynamics of asymptotically flat cosmological solutions

Thus, in order to determine the temperature and the chemical potentials in the microcanonical ensemble, we use the thermodynamic relations

$$\beta_C = \left(\frac{\partial S_C}{\partial \mathbb{M}}\right)\bigg|_{\mathbb{J},\mathbb{W}} = -T_C^{-1} = -n_{\mathfrak{m}}\frac{1}{2}\sqrt{\frac{\pi\alpha_2 k}{\mathcal{Z}}}\left(\frac{\mathcal{P}}{\mathcal{Z}}-\frac{\alpha_1}{\alpha_2}\right),$$

$$\Omega_C = -\beta_C^{-1}\left(\frac{\partial S_C}{\partial \mathbb{J}}\right)\bigg|_{\mathbb{M},\mathbb{W}} = -2\left(\frac{\mathcal{P}}{\mathcal{Z}}-\frac{\alpha_1}{\alpha_2}\right)^{-1},$$

$$\Phi_C = -\beta_C^{-1}\left(\frac{\partial S_C}{\partial \mathbb{W}}\right)\bigg|_{\mathbb{M},\mathbb{J}} = \frac{1}{4}\left(3\frac{\mathcal{P}}{\mathcal{Z}}+\frac{\alpha_1}{\alpha_2}\right)-\frac{\alpha_2\mathcal{J}-\alpha_0\mathcal{Z}}{\alpha_2\mathcal{P}-\alpha_1\mathcal{Z}},$$

where $\mathbb{M}=2\pi\mathcal{P}$, $\mathbb{J}=-2\pi\mathcal{J}$, and $\mathbb{W}=2\pi\mathcal{Z}$. Then, the first law of thermodynamics is found to fulfilled according to

$$\delta S_C = \beta_C(\delta\mathbb{M}-\Omega_C\delta\mathbb{J}-\Phi_C\delta\mathbb{W}),$$

such that the chemical potentials $\beta_C=\xi$, $\Omega_C=\mu/\xi$, and $\Phi_C=-\nu/\xi$ correspond to the conjugated to the mass, angular momentum and the additional spin-2 charge, respectively.

Outline

Maxwell CS gravity with torsion

Using the CS formalism, we presented the three-dimensional gravity theory based the deformed Maxwell algebra, [P. Concha, H.Safari (2019); H. Adami, P. Concha, E. Rodríguez, H. Safari (2020)]

$$[J_a, J_b] = \epsilon_{abc} J^c \qquad [J_a, P_b] = \epsilon_{abc} P^c$$
$$[J_a, Z_b] = \epsilon_{abc} Z^c \qquad [P_a, P_b] = \epsilon_{abc} (Z^c + \varepsilon P^c)$$

$\varepsilon \to 0^* \implies$ Maxwell algebra

(*) This limit can be interpreted as a vanishing cosmological constant limit, i.e. $\ell \to \infty$, when we set $\varepsilon = -2/\ell$. Considering the following redefinition of the generators,

$$L_a \equiv J_a - \varepsilon^{-1} P_a - \varepsilon^{-2} Z_a \quad ; \quad T_a \equiv \varepsilon^{-1} Z_a \quad ; \quad S_a \equiv \varepsilon^{-1} P_a + \varepsilon^{-2} Z_a \,,$$

the $\mathfrak{iso}(2,1) \otimes \mathfrak{so}(2,1)$ algebra is revealed,

$$[L_a, L_b] = \epsilon_{abc} L^c \,, \quad [L_a, T_b] = \epsilon_{abc} T^c \,, \qquad [S_a, S_b] = \epsilon_{abc} S^c \,.$$

Maxwell CS gravity with torsion

The gauge connection one-form A for the deformed Maxwell algebra reads

$$A = e^a P_a + \omega^a J_a + \sigma^a Z_a$$

The non-degenerate bilinear form of the deformed Maxwell algebra reads

$$\langle J_a J_b \rangle = \alpha_0 \eta_{ab} \,, \qquad \langle P_a P_b \rangle = (\varepsilon\alpha_1 + \alpha_2)\eta_{ab} \,,$$
$$\langle J_a P_b \rangle = \alpha_1 \eta_{ab} \,, \qquad \langle J_a Z_b \rangle = \alpha_2 \eta_{ab} \,,$$

where α_0, α_1 and α_2 are arbitrary constants.
The CS gravity action invariant under the deformed Maxwell algebra reads

$$I_{\mathfrak{tor}-\mathfrak{max}} = \frac{1}{16\pi G}\int_{\mathcal{M}} \alpha_0 \left(\omega^a d\omega_a + \frac{1}{3}\epsilon^{abc}\omega_a\omega_b\omega_c\right) + \alpha_1 \left(2R_a e^a + \frac{\varepsilon^2}{3}\epsilon^{abc}e_a e_b e_c + \varepsilon T^a e_a\right)$$
$$+ \alpha_2 \left(T^a e_a + 2R^a\sigma_a + \frac{\varepsilon}{3}\epsilon^{abc}e_a e_b e_c\right) \,,$$

$\varepsilon \to 0 \implies$ Maxwell CS gravity theory

Maxwell CS gravity with torsion

Imposing the conditions $\alpha_2 \neq 0$ and $\alpha_2 \neq -\varepsilon\alpha_1$, the e.o.m are given by the vanishing of the curvature two-forms

$$\begin{aligned} R^a &= 0\,, \\ T^a + \frac{\varepsilon}{2}\epsilon^{abc} e_b e_c &= 0\,, \\ D\sigma^a + \frac{1}{2}\epsilon^{abc} e_b e_c &= 0\,. \end{aligned}$$

As we can see, the field equations are those of Riemann-Cartan gravity (with zero curvature and constant torsion) plus the equation of motion involving the gravitational Maxwell field. Then, the geometries described by the e.o.m are non-Riemannian (with non-vanishing torsion) and locally flat.

Asymptotic conditions with chemical potentials

The radial dependence of the gauge field is eliminated asymptotically for an appropriate gauge choice, so that

$$A = g^{-1}dg + g^{-1}a_{(\varepsilon)}g \ ,$$

with $a_{(\varepsilon)} = a_t(t,\phi)dt + a_\phi(t,\phi)d\phi$. The asymptotic form of angular component a_ϕ is left invariant for the Lie-algebra-valued parameter

$$\lambda_{(\varepsilon)}[y,f,h] = \lambda_{(0)} + \frac{\varepsilon}{2}\mathcal{N} f P_0 \ ,$$

and providing that the functions $\mathcal{M}, \mathcal{N}$ and $\mathcal{F}$ now transform according to:

$$\begin{aligned} \delta_{(\varepsilon)}\mathcal{M} &= \delta_{(0)}\mathcal{M} \,, \\ \delta_{(\varepsilon)}\mathcal{N} &= \delta_{(0)}\mathcal{N} + \varepsilon\left(\mathcal{N}' f + 2\mathcal{N} f'\right) \,, \\ \delta_{(\varepsilon)}\mathcal{F} &= \delta_{(0)}\mathcal{F} \,, \end{aligned}$$

Thus, to preserve the asymptotic symmetries under evolution in time, the asymptotic form of the Lagrange multiplier a_t must to be of the form

$$a_t = \lambda_{(\varepsilon)}[\mu,\xi,\nu]$$

Asymptotic conditions with chemical potentials

As before, the asymptotic symmetry generators can be computed in the Regge-Teitelboim approach such that their variations reduce to

$$\delta Q[\lambda] = -\frac{k}{2\pi}\int_{\partial\Sigma} \langle \lambda \delta a \rangle \, d\phi \quad \Longrightarrow Q[y,f,h] = -\int d\phi \left(y\mathcal{J} + f\mathcal{P} + h\mathcal{Z} \right) ,$$

$$\mathcal{J} = \frac{k}{4\pi}\left(\alpha_2 \mathcal{F} + \alpha_1 \mathcal{N} + \alpha_0 \mathcal{M}\right) \quad ; \quad \mathcal{P} = \frac{k}{4\pi}\left[(\varepsilon\alpha_1 + \alpha_2)\mathcal{N} + \alpha_1 \mathcal{M}\right] \quad ; \quad \mathcal{Z} = \frac{k}{4\pi}\alpha_2 \mathcal{M}$$

The transformations law of the dynamical variables $\{\mathcal{J},\mathcal{P},\mathcal{Z}\}$ explicitly read

$$\begin{aligned}
\delta\mathcal{J} &= \mathcal{J}'y + 2\mathcal{J}y' + \mathcal{P}'f + 2\mathcal{P}f' + \mathcal{Z}'h + 2\mathcal{Z}h' - \frac{\alpha_0 k}{2\pi}y''' - \frac{\alpha_1 k}{2\pi}f''' - \frac{\alpha_2 k}{2\pi}h''' , \\
\delta\mathcal{P} &= \mathcal{P}'y + 2\mathcal{P}y' + (\mathcal{Z}' + \varepsilon\mathcal{P}')f + 2(\mathcal{Z} + \varepsilon\mathcal{P})f' - \frac{\alpha_1 k}{2\pi}y''' - \frac{(\alpha_2 + \varepsilon\alpha_1)k}{2\pi}f''' , \\
\delta\mathcal{Z} &= \mathcal{Z}'y + 2\mathcal{Z}y' - \frac{\alpha_2 k}{2\pi}y''' .
\end{aligned}$$

Asymptotic symmetries

We found the following **asymptotic symmetry** algebra for the **Maxwell CS gravity with torsion**: [H. Adami, P. Concha, E. Rodríguez, H. Safari (2020)]

$$
\begin{aligned}
i\,\{\mathcal{J}_m,\mathcal{J}_n\} &= (m-n)\,\mathcal{J}_{m+n} + c_{\mathcal{J}}\, m(m^2-1)\delta_{m+n,0}\,,\\
i\,\{\mathcal{J}_m,\mathcal{P}_n\} &= (m-n)\,\mathcal{P}_{m+n} + c_{\mathcal{P}}\, m(m^2-1)\delta_{m+n,0}\,,\\
i\,\{\mathcal{P}_m,\mathcal{P}_n\} &= (m-n)\,(\mathcal{Z}_{m+n}+\varepsilon\mathcal{P}_{m+n}) + (c_{\mathcal{Z}}+\varepsilon c_{\mathcal{P}})\; m(m^2-1)\delta_{m+n,0}\,,\\
i\,\{\mathcal{J}_m,\mathcal{Z}_n\} &= (m-n)\,\mathcal{Z}_{m+n} + c_{\mathcal{Z}}\, m(m^2-1)\delta_{m+n,0}\,.
\end{aligned}
$$

This algebra corresponds to an infinite-dimensional lift of the deformed Maxwell algebra.

$$\varepsilon \to 0 \implies \mathfrak{max}\text{-}\mathfrak{bms}_3 \text{ algebra}$$

Thermodynamics

In this case, the entropy reads

$$S = k\left[\alpha_0 \mu \mathcal{M} + \alpha_1 \left(\mu \mathcal{N} + \xi \mathcal{M} + \varepsilon \xi \mathcal{N}\right) + \alpha_2 \left(\nu \mathcal{M} + \xi \mathcal{N} + \mu \mathcal{F}\right)\right]_{\text{on-shell}} ,$$

In summary, the regularity conditions lead to the following chemical potentials

$$\mu = -\frac{2\pi n}{\sqrt{\mathcal{M}}} , \qquad \xi = \frac{2\pi}{\varepsilon}\left(\frac{n}{\sqrt{\mathcal{M}}} - \frac{m}{\sqrt{\mathcal{M} + \varepsilon \mathcal{N}}}\right) ,$$
$$\nu = \frac{2\pi n}{\sqrt{\mathcal{M}}}\left[\frac{1}{\varepsilon^2} + \frac{1}{2\mathcal{M}}\left(\mathcal{F} - \frac{\mathcal{N}}{\varepsilon}\right)\right] - \frac{2\pi m}{\varepsilon^2 \sqrt{\mathcal{M} + \varepsilon \mathcal{N}}} .$$

Then, in terms of the variables $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{F}$, the entropy reads

$$S_{\text{BH}} = 2\pi k \sqrt{\mathcal{M}}\left\{\left(\frac{\alpha_1 \varepsilon + \alpha_2}{\varepsilon^2}\right)\left(n + m\sqrt{1 + \frac{\varepsilon \mathcal{N}}{\mathcal{M}}}\right) - n\left[\alpha_0 + \frac{\alpha_2}{2\varepsilon}\left(\frac{\varepsilon \mathcal{F} - \mathcal{N}}{\mathcal{M}}\right)\right]\right\} .$$

It can be readily checked that, upon imposing the condition $m = n = -n_{\mathfrak{m}}$, the expression reduces to the entropy of the Maxwell cosmological solution in the limit $\varepsilon \to 0$.

Thermodynamics

In terms of the global charges $\mathcal{P}, \mathcal{J}$ y $\mathcal{Z}$, the entropy finally reads

$$S_{\text{BH}} = \frac{4\pi\sqrt{\pi k}\sqrt{\alpha_2}}{\varepsilon}\left[\frac{n}{2}\left(\frac{\mathcal{P}-\varepsilon\mathcal{J}}{\sqrt{\mathcal{Z}}}\right)+n\left(\frac{1}{\varepsilon}+\frac{\alpha_1-\varepsilon\alpha_0}{2\alpha_2}\right)\sqrt{\mathcal{Z}}-\frac{m}{\varepsilon}\sqrt{\frac{\alpha_2+\varepsilon\alpha_1}{\alpha_2}}\sqrt{\mathcal{Z}+\varepsilon\mathcal{P}}\right]$$

The thermodynamical quantities $\mathbb{M} = 2\pi\mathcal{P}$, $\mathbb{J} = -2\pi\mathcal{J}$, and $\mathbb{W} = 2\pi\mathcal{Z}$ satisfy the first law of thermodynamics,

$$\delta S_{\text{BH}} = \beta_{\text{BH}}(\delta\mathbb{M} - \Omega_{\text{BH}}\delta\mathbb{J} - \Phi_{\text{BH}}\delta\mathbb{W})\,,$$

such that the chemical potentials $\beta_{\text{BH}} = \xi$, $\Omega_{\text{BH}} = \mu/\xi$, and $\Phi_{\text{BH}} = -\nu/\xi$ correspond to the conjugated to the mass, angular momentum and the additional spin-2 charge, respectively.

We can express the black hole entropy in terms of the outer and inner horizons of the black hole

$$r_{\pm} = \frac{1}{\varepsilon}\left(\sqrt{\mathcal{M}} \pm \sqrt{\mathcal{M}+\varepsilon\mathcal{N}}\right)\,.$$

Thermodynamics

- $\varepsilon = -2/\ell$, $n = -m = 1$

$$S_{\text{out}} = 2\pi k \left(\frac{\alpha_0}{\ell} (r_+ - r_-) + \alpha_1 r_+ - \frac{\alpha_2 (r_+^2 - 2b)\ell}{2(r_+ - r_-)} \right) ,$$

S_{out} corresponds to a Maxwell extension (along α_2) of the entropy of the teleparallel black hole. The first law in this case takes the form

$$\delta\mathbb{M} = T_{\text{BH}} \delta S_{\text{out}} + \Omega_H \delta\mathbb{J} + \Phi \delta\mathbb{W}$$

where the extensive variables can be expressed as follows

$$T_{\text{BH}} = \frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+} , \qquad \Omega_H = \frac{1}{\ell} + \frac{r_-}{\ell r_+} , \qquad \Phi = \frac{\left(r_+^2 + 2b(1 + r_-/r_+) - 3r_+ r_-\right)\ell}{2(r_+ - r_-)^2} .$$

Outline

Comments and further developments

- We constructed generalized asymptotic conditions for three-dimensional Maxwell–Chern–Simons gravity, both in the torsionless and torsional cases.
- We found nontrivial solutions carrying mass, angular momentum, and an additional global spin-2 charge.
- We derived thermodynamic properties and a generalized entropy expression.
- It would be interesting to extend our study and the asymptotic symmetry analysis to the non-relativistic and ultra-relativistic regimes of the Maxwell CS gravity theory.
- We could also carry out the analysis of the energy bounds and asymptotic Killing spinors in the supersymmetric extensions of both Maxwell/Hietarinta and AdS–Lorentz gravity theories.

Thank you!