Universal self-gravitating skyrmions

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Based on:
A. V, JHEP **10**, 119 (2025) [arXiv:2503.03872 [hep-th]].

Motivation

- The Skyrme model is a non-linear effective field theory that represents the low energy limit of QCD at leading order in the 't Hooft expansion. *T. H. R. Skyrme, Nucl. Phys.* ***31****, 556-569 (1962).*

- When the Skyrme model is coupled to general relativity, it allows describing astrophysical objects supported by hadronic matter:

1. Topological black holes with Skyrme hair. *H. Luckock and I. Moss, Phys. Lett. B* ***176****, 341-345 (1986).*

2. The self-gravitating skyrmion: An exact solution of the Einstein $SU(2)$-Skyrme model, describing a topological soliton living in a 4-dimensional space-time in the presence of a cosmological constant. *E. Ayon-Beato, F. Canfora and J. Zanelli, Phys. Lett. B* ***752****, 201-205 (2016).*

- A more accurate description of hadrons at low energies requires considering higher order terms in the Skyrme theory and the inclusion of more flavors.

The Einstein $SU(N)$-Skyrme model

The Einstein $SU(N)$-Skyrme model in $D=4$ space-time dimensions is described by the action

$$I[g,U]=\int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa}+\frac{K}{4}\mathrm{Tr}[L^\mu L_\mu]+\frac{K\lambda}{32}\mathrm{Tr}\left[G_{\mu\nu}G^{\mu\nu}\right]\right) \ ,$$
$$L_\mu=U^{-1}\nabla_\mu U=L_\mu^a t_a \ , \quad G_{\mu\nu}=[L_\mu,L_\nu] \ ,$$

where $U(x)\in SU(N)$ and t_a are the generators of the $SU(N)$ Lie group.

The field equations of the system are given by

$$\nabla_\mu L^\mu+\frac{\lambda}{4}\nabla_\mu[L_\nu,G^{\mu\nu}]=0 \ ,$$
$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=\kappa T_{\mu\nu} \ ,$$

where $T_{\mu\nu}$ is the energy-momentum tensor describing the hadronic matter content.

The topological charge (baryon number) is defined by

$$B=\frac{1}{24\pi^2}\int_\Sigma \rho_{\mathrm{B}}\,\mathrm{d}V \ , \qquad \rho_{\mathrm{B}}=\epsilon^{ijk}\mathrm{Tr}\Big[\left(U^{-1}\partial_i U\right)\left(U^{-1}\partial_j U\right)\left(U^{-1}\partial_k U\right)\Big] \ .$$

The self-gravitating skyrmion: A short review

- The self-gravitating skyrmion is an exact solution of the Einstein $SU(2)$-Skyrme model with unit baryon number. (See: *N. S. Manton and P. J. Ruback, Phys. Lett. B* ***181****, 137-140 (1986); E. Ayon-Beato, F. Canfora and J. Zanelli, Phys. Lett. B* ***752****, 201-205 (2016); E. Ayón-Beato, F. Canfora, M. Lagos, J. Oliva and A. Vera, Eur. Phys. J. C* ***80****, no.5, 384 (2020).*)

The usual way to write this solution is using the exponential parametrization

$$U(x^{\mu}) = \cos(\alpha)\mathbf{1}_{2\times 2} + \sin(\alpha)n^{a}\tau_{a} \ ,$$
$$n^{1} = \sin\Theta\cos\Phi \ , \quad n^{2} = \sin\Theta\sin\Phi \ , \quad n^{3} = \cos\Theta \ ,$$

where $\alpha = \alpha(x^{\mu})$, $\Theta = \Theta(x^{\mu})$, $\Phi = \Phi(x^{\mu})$. The isospin vectors satisfy $n^{a}n_{a} = 1$, with $\tau_{a} = i\sigma_{a}$, being σ_{a} the Pauli matrices.

When the three degrees of freedom are chosen as

$$\Phi = \frac{\gamma+\phi}{2} \ , \quad \tan(\Theta) = \frac{\cot(\frac{\theta}{2})}{\cos(\frac{\gamma-\phi}{2})} \ , \quad \tan(\alpha) = \frac{\sqrt{1+\tan^{2}\Theta}}{\tan(\frac{\gamma-\phi}{2})} \ ,$$

and the space-time metric has the form

$$ds^{2} = -dt^{2} + h(t)[(d\gamma + \cos(\theta)d\varphi)^{2} + d\theta^{2} + \sin^{2}(\theta)d\varphi^{2}] \ ,$$

the Skyrme equations are automatically satisfied!

The self-gravitating skyrmion: A short review

The Einstein equations are reduced to the following system:

$$(\dot{h})^2 - \frac{\Lambda}{3}h^2 - \frac{K\kappa\lambda}{32}\frac{1}{h^2} - \frac{K\kappa - 2}{8} = 0 \ ,$$
$$\ddot{h} - \frac{\Lambda}{3}h + \frac{K\kappa\lambda}{32}\frac{1}{h^3} = 0 \ ,$$

which can be solved analytically. *F. Canfora, A. Paliathanasis, T. Taves and J. Zanelli, Phys. Rev. D **95**, no.6, 065032 (2017).*

In the static case, $h(t) = h_0$, the system is solved by a fine-tuning between the coupling constants

$$h_0^2 = \frac{3(2 - K\kappa)}{4\Lambda} \ , \qquad \lambda = \frac{3(2 - K\kappa)^2}{8K\kappa\Lambda} \ .$$

The above configurations describe a topological soliton with unit baryon charge

$$\rho_{\mathrm{B}} = \frac{3}{2}\sin(\theta) \quad \Rightarrow \quad B = \frac{1}{24\pi^2}\int \rho_{\mathrm{B}}\, \mathrm{d}\gamma \mathrm{d}\theta \mathrm{d}\varphi = 1 \ .$$

- In the static case, this a self-gravitating skyrmion in a space-time with geometry $\mathbb{R} \times S^3$, and where the cosmological constant and the scale factor are fixed in terms of the coupling constants of the model.
- In the dynamical case this is a bouncing state in which the universe contracts and expands, and where a minimum size of the universe exists bounded by the skyrmion scale.

From the exponential parametrization to the Euler angles parametrization

- The U field that allows the existence of the self-gravitating skyrmion can be easily written using the Euler angles parametrization.

An arbitrary element of $SU(2)$ can be expressed as

$$U(x^{\mu}) = e^{F_1 \cdot \tau_3} e^{F_2 \cdot \tau_2} e^{F_3 \cdot \tau_3} \ ,$$

where $F_1 = F_1(x^{\mu})$, $F_2 = F_2(x^{\mu})$, $F_3 = F_3(x^{\mu})$.

The matter field of the self-gravitating skyrmion can be obtained simply choosing the degrees of freedom as

$$F_1(x^{\mu}) = \gamma \ , \qquad F_2(x^{\mu}) = \theta \ , \qquad F_3(x^{\mu}) = \varphi \ .$$

The above parametrization is very convenient not only from the computation point of view, it also provides a natural way to generalized the self-gravitating skyrmion to include arbitrary number of flavors.

Arbitrary flavors and high baryonic charge

- The maximal embedding Ansatz of $SU(2)$ into $SU(N)$ allows working with arbitrary flavors. This uses as a base three $N \times N$ matrices, which give rise to an irreducible representation of $SU(2)$ into $SU(N)$. *S. Bertini, S. L. Cacciatori and B. L. Cerchiai, J. Math. Phys. 47, 043510 (2006).*

The matter field $U(x) \in SU(N)$ in the Euler angles parametrization reads

$$U = e^{F_1(x^\mu)\cdot T_3} e^{F_2(x^\mu)\cdot T_2} e^{F_3(x^\mu)\cdot T_3} ,$$

where T_i are the generators of a 3-dimensional sub-algebra of $\mathfrak{su}(N)$. Explicitly,

$$T_1 = -\frac{i}{2}\sum_{j=2}^{N}\sqrt{(j-1)(N-j+1)}(E_{j-1,j}+E_{j,j-1}) ,$$

$$T_2 = \frac{1}{2}\sum_{j=2}^{N}\sqrt{(j-1)(N-j+1)}(E_{j-1,j}-E_{j,j-1}) ,$$

$$T_3 = i\sum_{j=1}^{N}(\frac{N+1}{2}-j)E_{j,j} ,$$

where $(E_{i,j})_{mn} = \delta_{im}\delta_{jn}$, and being δ_{ij} the Kronecker delta. These matrices satisfy

$$[T_a, T_b] = \epsilon_{abc}T_c .$$

The trace of the generators depends explicitly on the N parameter.

$$\mathrm{Tr}\,(T_b T_c) = -\frac{1}{2}a_N\delta_{bc} , \qquad a_N = \frac{N\left(N^2-1\right)}{6} .$$

For different values of the flavor number the matter field leads to different kind of solutions.

Arbitrary flavors and high baryonic charge

By considering the Skyrme field using the maximal embedding Ansatz of $SU(2)$ into $SU(N)$

$$U(x^\mu) = e^{F_1(x^\mu)\cdot T_3} e^{F_2(x^\mu)\cdot T_2} e^{F_3(x^\mu)\cdot T_3} \ ,$$
$$F_1(x^\mu) = \gamma \ , \qquad F_2(x^\mu) = \theta \ , \qquad F_3(x^\mu) = \varphi \ ,$$

we found that the Skyrme equations are still automatically satisfied, while the Einstein equations are reduced to the following system for the scale factor:

$$(\dot{h})^2 - \frac{\Lambda}{3}h^2 - \frac{K\kappa a_N \lambda}{2}\frac{1}{h^2} - \frac{K\kappa a_N}{2} + 1 = 0 \ ,$$
$$\ddot{h} - \frac{\Lambda}{3}h + \frac{K\kappa a_N \lambda}{2}\frac{1}{h^3} = 0 \ .$$

In the static case, $h(t) = h_0$, the system is solved by the following fine-tuning

$$h_0^2 = \frac{3}{2\Lambda}\left(1 - \frac{1}{2}a_N K\kappa\right) \ , \qquad \lambda = \frac{3}{2\Lambda a_N K\kappa}\left(1 - \frac{1}{2}a_N K\kappa\right)^2 \ .$$

These are the generalization of $SU(2)$ self-gravitating skyrmion now including arbitrary flavors. The main relevance of these novel solutions is that the topological charge is not a unit. Instead, it directly depends on the number of flavors

$$\rho_{\rm B} = \frac{3}{2}a_N \sin(\theta) \quad \Rightarrow \quad B = \frac{1}{24\pi^2}\int \rho_{\rm B}\, d\gamma d\theta d\varphi = a_N \ .$$

For the first allowed number of flavors; $N = \{2, 3, 4, \dots\}$, we obtain $B = \{1, 4, 10, \dots\}$.

The generalized Skyrme model

- An optimal description of the low-energy sector of QCD requires considering higher-order terms that come from the 't Hooft expansion.
- In the context of the Skyrme model, this implies supplementing the action with higher-order derivative terms. *L. Marleau, Phys. Rev. D **45**, 1776-1781 (1992).*

The first two sub-leading terms (after the Skyrme term) are given by

$$\begin{aligned}
\mathcal{L}_6 &= \frac{c_6}{96} \mathrm{Tr}\left[G_\mu{}^\nu G_\nu{}^\rho G_\rho{}^\mu \right] \ , \\
\mathcal{L}_8 &= -\frac{c_8}{256} \left(\mathrm{Tr}\left[G_\mu{}^\nu G_\nu{}^\rho G_\rho{}^\sigma G_\sigma{}^\mu \right] - \mathrm{Tr}\left[\{ G_\mu{}^\nu , G_\rho{}^\sigma \} G_\nu{}^\rho G_\sigma{}^\mu \right] \right) .
\end{aligned}$$

The field equations of the generalized Skyrme model are

$$\begin{aligned}
&\frac{K}{2}\left(\nabla^\mu L_\mu + \frac{\lambda}{4} \nabla^\mu [L^\nu, G_{\mu\nu}] \right) + 3c_6 [L_\mu, \nabla_\nu [G^{\rho\nu}, G_\rho{}^\mu]] \\
&+4c_8 \left[L_\mu, \nabla_\nu \left(G^{\nu\rho} G_{\rho\sigma} G^{\sigma\mu} + G^{\mu\rho} G_{\rho\sigma} G^{\nu\sigma} + \{ G_{\rho\sigma}, \{ G^{\mu\rho}, G^{\nu\sigma} \} \} \right) \right] \ = \ 0 \ .
\end{aligned}$$

Also, the Einstein equations must be considered including the contributions to the energy-momentum tensor that comes from the new terms.

Generalized self-gravitating skyrmions

Considering again the same Ansatz for the matter field as well as for the space-time metric, the generalized Skyrme equations are reduced to the following constraint

$$c_8 = \frac{1}{(N^2-5)} c_6 h_0^2 \ .$$

The energy-density of this solution is given by

$$\varepsilon = \frac{3}{2} K a_N \frac{(h_0^2+\lambda)}{h_0^4} + \frac{1}{4} c_6 a_N \frac{1}{h_0^6} \ .$$

Some relevant facts of these solutions are:

- The Einstein equations reduce to two algebraic constraints involving all the constants of the theory.
- The sign of the coupling constants changes for different values of N. As expected, c_8 is a small number compared to c_6, and the difference becomes larger as N grows.
- The energy-density is positive definite, at least for any $c_6 > 0$.

Skyrmions at finite volume

- From the matter field that describes the self-gravitating skyrmions it is possible to construct skyrmion states in flat space-time at a finite volume. *P. D. Alvarez, F. Canfora, N. Dimakis and A. Paliathanasis, Phys. Lett. B* ***773****, 401-407 (2017).*

Let us consider the $U(x)$ field including a soliton profile

$$U(x^\mu) = e^{F_1(x^\mu)\cdot T_3} e^{F_2(x^\mu)\cdot T_2} e^{F_3(x^\mu)\cdot T_3} ,$$
$$F_1(x^\mu) = x , \qquad F_2(x^\mu) = H(y) , \qquad F_3(x^\mu) = z ,$$

in a space-time metric of a box

$$ds^2 = -dt^2 + l^2(dx^2 + dy^2 + dz^2) .$$

For simplicity, we take $N = 2$.

With this Ansatz the complete set of field equations is reduced to a single differential equation

$$(H')^2 + \frac{\lambda}{4(2l^2+\lambda)}\cos(2H) + \frac{c_6}{4Kl^2(2l^2+\lambda)}\sin^2(H)(H')^2$$
$$+ \frac{c_8}{512Kl^4(2l^2+\lambda)}\{\cos(4H) - 4\cos(2H) + 32\sin^2(H)(H')^2 - 96\cos^2(H)(H')^4\} = E_0 .$$

The same value of the topological charge is obtained when the following boundary conditions are imposed: $H(2\pi) = \pi/2$, $H(0) = 0$.

- The present solution describes multi-skyrmions states at a finite volume and arbitrary flavors.

Summary

The self-gravitating skyrmion can be generalized to include:

- Arbitrary flavors and, consequently, high baryon number.
- Higher order correction in 't Hooft expansion

These results open the possibility to study compact stars, inhomogeneous condensates, wormholes solutions, among others, using analytical methods and considering high baryon number without any approximation.