Spontaneous Lorentz Symmetry Breaking and Cosmological Gravitational Particle Production

Simón Riquelme M.
USS

UNIVERSIDAD SAN SEBASTIAN

CECs
CENTRO DE ESTUDIOS CIENTÍFICOS

Departamento de Física
Facultad de Ciencias Físicas y Matemáticas,
Universidad de Chile, Santiago, Chile

Work done in collaboration with Pablo Pais & Mauricio Valenzuela

arxiv 25xx.xxxxx

Harmonic Oscillator

$$S = \int dt\, L \,, \quad L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x \,, \quad \omega^2 = k/m$$

$$\overset{EL}{\Rightarrow} \quad m\ddot{x} = -kx \,.$$

To quantize $\quad p \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x} \,, \quad (x, p) \to (\hat{x}, \hat{p})$

s.t. $\quad [\hat{x}, \hat{p}] = i\hbar \,,$

$\hat{x}^\dagger = \hat{x} \,, \quad \hat{p}^\dagger = \hat{p} \,.$

Bold move $\Rightarrow \quad V(x) \to V(x, t)$

Here $\quad \omega \to \omega(t)$

EOM $\quad \ddot{x} + \omega^2(t)\, x = 0 \quad$ (Dropped hats)

Now we trade $(x(0), \dot{x}(0)) \to a$ s.t. $a^\dagger \neq a$,
and write
$$x(t) = f(t)a + f^*(t)a^\dagger,$$
$$\text{with } \ddot{f} + w^2(t) f = 0.$$

Imposing
$$[x(t), \dot{x}(t)] = \frac{i\hbar}{m} \Rightarrow \langle f, f \rangle [a, a^\dagger] = 1,$$

where
$$\langle f, g \rangle \equiv \left(\frac{im}{\hbar}\right) \left\{ f^* \dot{g} - \dot{f}^* g \right\} \quad \left(\begin{array}{c}\text{This is time-ind.}\\ \text{due to EOM}\end{array}\right)$$

Now assume $\langle f, f \rangle > 0$ then wlog $\langle f, f \rangle = 1$
$$\Rightarrow [a, a^\dagger] = 1.$$

Moreover, $a = \langle f, x \rangle$, $a^\dagger = -\langle f^*, x \rangle$. (Time-ind.)

Introduce $|0\rangle$ s.t. $a|0\rangle = 0$, then

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \quad N|n\rangle = n|n\rangle, \quad N \equiv a^\dagger a.$$

Notice: $f(t)$ arbitrary up to $\langle f, f\rangle = 1$

f and a may change keeping x unchanged

SPECIAL CASE: $\omega(t) = \omega$ constant $\Rightarrow$ Energy is conserved

$\Rightarrow$ A special choice of $f(t)$ is selected if $|0\rangle$ is required to be the ground state of the Hamiltonian.

Indeed,

$$H = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m \omega^2 x^2$$

$$= \tfrac{1}{2} m \left[(\dot{f}^2 + \omega^2 f^2) a a + (\dot{f}^2 + \omega^2 f^2)^* a^\dagger a^\dagger + (|\dot{f}|^2 + \omega^2 |f|^2)(a a^\dagger + a^\dagger a) \right],$$

$[a, a^\dagger] = 1$

$$\Rightarrow \quad H|0\rangle = \frac{1}{2} m (\dot{f}^2 + \omega^2 f^2)^* a^\dagger a^\dagger |0\rangle + (|\dot{f}|^2 + \omega^2 |f|^2)|0\rangle.$$

$$H|0\rangle = \frac{1}{2}m(\dot{f}^2 + w^2 f^2)^* a^\dagger a^\dagger |0\rangle + (|\dot{f}|^2 + w^2|f|^2)|0\rangle .$$

$|0\rangle$ eigenstate of H $\Rightarrow$ $\dot{f}^2 + w^2 f^2 = 0 \Rightarrow \dot{f} = \pm i w f$

$$\Rightarrow \langle f, f\rangle = \mp \frac{2mw}{\hbar}|f|^2$$

Recalling $\langle f, f\rangle = 1$,

$$f(t) = \sqrt{\frac{\hbar}{2mw}}\exp(-iwt)$$

POSITIVE FREQUENCY SOLUTION

With this choice,

$$H = \frac{1}{2}\hbar w\,(a a^\dagger + a^\dagger a) = \hbar w(N + 1/2),$$

$$H|0\rangle = \frac{1}{2}\hbar w|0\rangle, \quad \langle 0|x|0\rangle = 0, \quad \langle 0|x^2|0\rangle = \frac{\hbar}{2mw}$$

Zero-point fluctuations

Let us come back to

$$\ddot{x} + w^2(t)\,x = 0.$$

We will assume that

$$w(t) \xrightarrow[t\to-\infty]{} w_{in}\,, \quad w(t) \xrightarrow[t\to+\infty]{} w_{out}\,, \quad (w_{in}, w_{out}) \text{ constants.}$$

QUESTION: If the oscillator starts in $|0_{in}\rangle$ appropriate to w_{in} as $t\to-\infty$, what is the state as $t\to+\infty$?

In Heisenberg picture this is equivalent to ask:

How is $|0_{in}\rangle$ expressed as a Fock state in the out-Hilbert space appropriate to w_{out}?

We need to relate $(a_{in}, a^\dagger_{in})$ to $(a_{out}, a^\dagger_{out})$ associated with $(f_{in}(t), f_{out}(t))$, whose asymptotic behavior is

$$f_{in,out} \xrightarrow[t\to\mp\infty]{} \sqrt{\frac{\hbar}{2m w_{in,out}}}\; \exp\left(-i\, w_{in,out}\, t\right)$$

MATHEMATICAL FACT: SECOND ORDER EOM $\Rightarrow$ ADMITS A TWO-PARAMETER FAMILY OF SOLUTIONS

$\Rightarrow$ THERE MUST EXIST COMPLEX CONSTANTS (α, β) SUCH THAT

$$f_{out} = \alpha f_{in} + \beta f_{out}^{*}.$$

Normalization $\langle f_{out}, f_{out} \rangle = 1 \Rightarrow |\alpha|^2 - |\beta|^2 = 1.$

Moreover,

$$a_{out} = \langle f_{out}, x \rangle = \langle \alpha f_{in} + \beta f_{in}^{*}, x \rangle$$
$$= \alpha a_{in} - \beta^{*} a_{in}^{\dagger}.$$

This is a so-called "Bogoliubov transformation", with (α, β) Bogoliubov coefficients.

Notice now that

$$\langle 0_{in} | N_{out} | 0_{in} \rangle = \langle 0 | a_{out}^{\dagger} a_{out} | 0 \rangle = |\beta|^2 \neq 0.$$

THE TIME DEPENDENCE OF $\omega(t)$ EXCITES THE OSCILLATOR, AND $|\beta|^2$ CHARACTERIZES THE EXCITATION NUMBER.

LET US MENTION TWO EXTREME CASES:
"ADIABATIC" AND "SUDDEN"

Adiabatic: $\dot{w}/w \ll w$

Frequency changes very slowly compared to the period of oscillation.

$\Rightarrow$ Almost no excitation,
$|\beta| \ll 1.$

Generically, $\beta \sim \exp(-w_0 T)$, w_0 some typical freq., T time scale for the variations of the freq.

$$\Rightarrow \quad f_{t_0} = \sqrt{\frac{\hbar}{2mw(t_0)}}\,, \quad \dot{f}_{t_0}(t_0) = -i w(t_0) f_{t_0}(t_0)$$

INSTANTANEOUS POSITIVE FREQUENCY SOLUTION
("Lowest order adiabatic ground state at t_0").

Higher order adiabatic ground state may be found through a function of the "WKB form"

$$\left(\frac{\hbar}{2mW(t)}\right)^{1/2} \exp\left(-i\int^t W(t')\,dt'\right), \quad \mathcal{D}^2 W(t) = 0 \Rightarrow W(t) = w(t) + \dots$$

some 2nd order diff. op.

Sudden transitions:

W changes instantaneously from W_{in} to W_{out} at some time t_0. For $t_0 = 0$ one finds

$$\alpha = \frac{1}{2}\left(\sqrt{\frac{W_{in}}{W_{out}}} + \sqrt{\frac{W_{out}}{W_{in}}}\right), \quad \beta = \frac{1}{2}\left(\sqrt{\frac{W_{in}}{W_{out}}} - \sqrt{\frac{W_{out}}{W_{in}}}\right).$$

Numerical example: $W_{out} = 4 W_{in} \Rightarrow \alpha = 5/4,\ \beta = -3/4,$

$$\Rightarrow \langle 0_{in} | N_{out} | 0_{in} \rangle = |\beta|^2 = 9/16$$

About "half an excitation".

This is all super nice, however "there is an elephant in the room" as the gringos say:

$W = W(t)$? WHO ORDERED THAT???

We want to consider a self-contained system that dinamically delivers such a possibility. Let us build a toy model which does the job in a very non-trivial way that resembles our field theory constructions we will soon unveil.

TOY MODEL

Consider two coupled 0+1 D dof, a "collective coordinate" $\lambda(t)$ moving in a double-well and a "harmonic probe" $x(t)$. The dynamics are governed by

$$L[\lambda, x] = \frac{1}{2}\mu\dot{\lambda}^2 + \frac{1}{2}m\dot{x}^2 - \frac{\eta}{4}(\lambda^2 - v^2)^2 - \frac{1}{2}k x^2 + \frac{g}{2}\dot{\lambda}^2 x,$$

$\{\mu, m, k, \eta, g\} > 0.$

In the decoupled limit $g=0$,

TWO STATIC VACUA @ $\lambda = \pm v$, $x = 0$.

In this regime, small fluctuations have characteristic freq.

$$\omega_\lambda^2 = \frac{2\eta v^2}{\mu}, \quad \omega_x^2 = \frac{k}{m}.$$

EOM

$$(\mu + gx)\ddot{\lambda} + g\dot{\lambda}\dot{x} + \eta\lambda(\lambda^2 - v^2) = 0$$

$$m\ddot{x} + kx = \frac{g}{2}\dot{\lambda}^2$$

EOM

$$(\mu + g x)\ddot{\lambda} + g\dot{\lambda}\dot{x} + \eta\lambda(\lambda^2 - v^2) = 0$$

$$m\ddot{x} + kx = \frac{g}{2}\dot{\lambda}^2$$

Hopeless

Way to go? Perturbatively in g.

Consider $\mu\ddot{\lambda}_0 + \eta\lambda_0(\lambda_0^2 - v^2) = 0$

$/\cdot\dot{\lambda}\ /\int dt$

$$\Rightarrow \quad \frac{1}{2}\mu\dot{\lambda}_0^2 + \frac{\eta}{4}(\lambda_0^2 - v^2)^2 = E \quad , \quad E \geq 0$$

Conserved energy

Define $\lambda_\pm^2 = v^2 \pm 2\sqrt{\frac{E}{\eta}}$ "Turning Points"

$$\Rightarrow \quad \dot{\lambda}_0^2 = \frac{\eta}{2\mu}(\lambda_+^2 - \lambda_0^2)(\lambda_0^2 - \lambda_-^2)$$

As in the Kepler problem the dynamics depends crucially in the value of E.

- $0 < E < \eta v^4/4$ Single Well (oscillations around the minima)

$$\lambda_0(t) = \pm \lambda_+ \underbrace{\mathrm{dn}(\Omega(t-t_0)|n)}_{\text{Jacobi Elliptic functions}}$$

with period

$$T(E) = \frac{2K(n)}{\Omega}$$

with freq. $\Omega \equiv \sqrt{\frac{\eta}{2n}}\,\lambda_+$,

and "modulus" $n = 1 - \frac{\lambda_-^2}{\lambda_+^2}$, $n \in [0,1]$

When $E \ll \eta v^4/4$

$$\lambda_0(t) \simeq \pm v + A\cos(\omega_\lambda t + \phi)\,, \quad \omega_\lambda^2 = \frac{2\eta v^2}{n}$$

- $E = \eta v^4/4$, "Separatrix" (hyperbolic profile)

$\lambda_- = 0$, $n \to 1$, $\lambda_0(t) = \pm\sqrt{2}\, v\, \mathrm{sech}\left(\sqrt{\frac{\eta}{n}}\, v(t-t_0)\right)$

with $T \to \infty$.

• $E > \eta v^4/4$ Double Well (over-the-barrier motion)

$$\lambda_0(t) = \pm \lambda_+ \, \mathrm{cn}\left(\tilde{\Omega}(t-t_0) \,|\, \tilde{m}\right), \quad \tilde{\Omega} = \sqrt{\frac{\eta}{2m}} \sqrt{\lambda_+^2 - \lambda_-^2},$$

Jacobi Elliptic function

$$-\lambda_-^2 > 0, \quad \tilde{T} = \frac{4K(\tilde{m})}{\tilde{\Omega}}.$$

Among these possibilities:

1) Over-the-barrier: Needs "Floquet analysis" with quasi-energies rather than dispersive vacua. Does not offer a clear in/out setup.

2) Separatrix: Asymptotic "tachyon".

3) Single Well: Does the job and, moreover, naturally produces a brief "tachyonic window" that seeds particle creation through kinetic mixing.

Given the background $\lambda_0(t)$, the probe satisfies

$$\ddot{x}_0 + \omega_x^2 x_0 = \frac{g}{2m} \dot{\lambda}_0^2 \quad , \quad \omega_x^2 = \frac{k}{m} \, ,$$

$$\Rightarrow \quad x_0(t) = \frac{g}{2m\omega_x} \int_{t_0}^{t} ds \, \sin\left(\omega_x (t-s)\right) \dot{\lambda}_0^2(s) \, , \quad \begin{array}{l} x_0(t_0) = 0 \\ \dot{x}_0(t_0) = 0 \end{array}$$

Background of x backreacts at $O(g)$.

Now we expand, $\lambda(t) = \lambda_0(t) + \delta\lambda(t)$, $x(t) = x_0(t) + \delta x(t)$.

The quadratic Lagrangian of fluctuations @ $O(g)$ is then given by

$$L^{(2)} = \frac{1}{2}\mu \, \delta\dot{\lambda}^2 + \frac{1}{2} m \, \delta\dot{x}^2 - \frac{1}{2} K \, \delta\lambda^2 - \frac{1}{2} k \, \delta x^2 + g \dot{\lambda}_0 \, \delta\dot{\lambda} \, \delta x \, ,$$

$$K \equiv \eta \left(3\lambda_0^2 - v^2\right) \quad , \quad \Omega_\lambda^2 \equiv \frac{K}{\mu} \, , \quad \omega_x^2 = \frac{k}{m} \, .$$

EOM

$$\mu\,\delta\ddot{\lambda} + \kappa\,\delta\lambda = -g\,\ddot{\lambda}_0\,\delta X - g\,\dot{\lambda}_0\,\delta\dot{X}$$

$$m\,\delta\ddot{X} + K\,\delta X = g\,\dot{\lambda}_0\,\delta\dot{\lambda}$$

Harmonic Ansatz

$$\delta\lambda(t) = \Lambda\exp(-i\omega t), \quad \delta X(t) = \mathcal{X}\exp(-i\omega t)$$

$$\Rightarrow \begin{pmatrix} \kappa-\mu\omega^2 & g(\ddot{\lambda}_0 - i\omega\dot{\lambda}_0) \\ ig\omega\dot{\lambda}_0 & K-m\omega^2 \end{pmatrix}\begin{pmatrix}\Lambda \\ \mathcal{X}\end{pmatrix} = 0$$

Non-trivial solution

$$\Rightarrow (\kappa-\mu\omega^2)(K-m\omega^2) - g^2(\omega^2\dot{\lambda}_0^2 + \underbrace{i\omega\dot{\lambda}_0\ddot{\lambda}_0}) = 0$$

Naive frequency-domain problem is non-Hermitian.

$\Rightarrow$ Energy can flow between the system and the background, which is precisely the mechanism behind particle creation.

Way out: Hamiltonian normal-mode problem is manifestly real and returns real ω^2.

In short,

$$\omega_\pm^2 = \frac{1}{2m\mu}\left(mK + \mu k + g^2\dot{\lambda}_0^2 \pm \sqrt{(mK+\mu k+g^2\dot{\lambda}_0^2)^2 - 4m\mu k K} \right)$$

$$K = \eta(3\lambda_0^2 - v^2)$$

Notice that if $g=0$,

$$\omega_+^2 = \max\{\omega_\lambda^2, \omega_x^2\},\ \omega_- = \min\{\omega_\lambda^2, \omega_x^2\}.$$

To ensure healthy in/out modes while retaining a finite tachyonic interval we take λ_0 to be the single-well solution with energy $\eta v^4/9 < E < \eta v^4/4$.

With this choice $K>0$ for most of each cycle, whereas the motion transiently enters $|\lambda_0| < v/\sqrt{3}$ near the inner turning point producing a brief $K<0$ window and controlled particle creation via kinetic mixing.

To calculate particle production we seek for instantaneous normal coordinates

$$(Q_\pm, P_\pm).$$

The quadratic Hamiltonian then reads

$$H^{(2)} = \frac{1}{2}\sum_{a=\pm}\left(P_a^2 + W_a^2(t) Q_a^2\right) + \frac{1}{2}\Sigma(t)\, Q_+ Q_- ,$$

$$\Sigma(t) \equiv -\frac{g\,\ddot{\lambda}_0(t)}{2\sqrt{m\mu}} \cos\left(\frac{g\lambda_0(t)}{\sqrt{m\mu}}\right).$$

Negligible on the adiabatic in/out slices used to define particle number.

We now quantize, $[Q_a, P_b] = i\delta_{ab}$ $(\hbar = 1)$.

Fix two adiabatic times

t_{in} (before the first tachyonic interval of W_-^2)

and t_{out} (after the second interval, before the next period)

and define
$$a_a^{in/out} = \sqrt{\frac{W_a^{in/out}}{2}}\, Q_a(t_{in/out}) + \frac{i}{\sqrt{2W_a^{in/out}}} P_a(t_{in/out}),$$

$$W_a^{in/out} = W_a(t_{in/out}),\quad \text{such that } a_a^{in}|0_{in}\rangle = 0.$$

Linearity of the evolution implies the Bogoliubov relations

$$a_a^{out} = \sum_{b=\pm} \left(\alpha_{ab}\, a_b^{in} + \beta_{ab}\, a_b^{in\dagger} \right), \quad \sum_c \left(\alpha_{ac} \alpha_{bc}^* - \beta_{ac} \beta_{bc}^* \right) = \delta_{ab}.$$

The produced quanta in branch a are then

$$n_a \equiv \langle 0_{in} | a_a^{out\dagger} a_a^{out} | 0 \rangle = \sum_b |\beta_{ab}|^2 .$$

I spare you the details of the calculation, which is quite technical. One can show that

$$n_-^{(one\ cycle)} \simeq \sinh^2 \left(2 \int_{t_1}^{t_2} dt \sqrt{-W_-^2(t)} \right),$$

$$n_+^{(one\ cycle)} = |\beta_+|^2 ,$$

$$\beta_+(t_{out}) \simeq \int_{t_{in}}^{t_{out}} dt\, \frac{\dot{W}_+}{2W_+} \exp\left(-2i \int^t W_+ \right) - \frac{1}{4} \int_{t_{in}}^{t_{out}} dt\, \frac{g\, \dot{\lambda}_0}{\sqrt{\mu m}} .$$

It so happens that $\quad n_+^{(one\ cycle)} \ll n_-^{(one\ cycle)}$

$\Big(N_\lambda \simeq n_-^{(one\ cycle)}$ & N_x parametrically subleading in the small-g regime $\Big)$

PROCA THEORY AND CGPP

Consider

$$\sqrt{-g}\,\mathcal{L}[A,\varphi] = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}a^2 m^2 A_\mu A^\mu - \tfrac{1}{2}a^2 \partial_\mu\varphi\,\partial^\mu\varphi$$
$$- \tfrac{1}{2}a^4 M^2\varphi^2 + \text{MIXING OPERATOR}(A_\mu,\varphi)$$

- Lorentzian possibility : $-\lambda a^4 A_\mu \partial^\mu \varphi$

  $\sim \rho\, \Pi_c \Box \varphi$, Π_c : long. mode of A^μ

Here $\rho \equiv \lambda/m$.

In short, $W_+^2 = |\vec{p}^2| + \underbrace{\dfrac{a^2M^2}{1-\rho^2}}_{\text{Boring}} - \dfrac{a''}{a}$, $W_-^2 = |\vec{p}|^2 - \dfrac{a''}{a}$

- Cosmological Lorentz-breaking "arrow of time".

An inflationary background driven by a scalar field ϕ with homogeneous background $\phi_0 = \phi_0(t)$ defines a preferred foliation of constant inflaton surfaces, inducing a unit normal vector field

$$n_\mu \equiv -\frac{\partial_\mu \phi}{\sqrt{-g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi}} = -a\,\delta^0_\mu\,.$$

→ Logical possibility

$$A_\mu \partial^\mu \varphi \rightarrow g^{\mu\nu} g^{\rho\sigma} n_\mu n_\rho A_\nu \partial_\sigma \varphi$$

lead to a superluminal mode. DEAD END.

Winning Possibility

$$n^\mu A_\mu \varphi \sim \dot{\Pi} \varphi$$

leads to

$$\omega_\pm^2 = |\vec{P}|^2 + \frac{a^2 M^2 c_s^{-2}}{2} - (1-\varepsilon)\mathcal{H}^2$$

$$+ \left\{ \left(|\vec{P}|^2 + \frac{a^2 M^2 c_s^{-2}}{2} \right)^2 - |\vec{P}|^2 (|\vec{P}|^2 + a^2 M^2) + \varepsilon M^2 (c_s^{-2} - 1) a^2 \mathcal{H}^2 \right\}^{1/2}$$

$$c_s \equiv \frac{M^2}{M^2 + \eta^2}, \quad \mathcal{H} = a'/a, \quad \varepsilon = -\mathcal{H}'/\mathcal{H}^2.$$

$$\eta \equiv K/m$$

STAY TUNED!

GRACIAS!