Partial decidability protocol for the Wang tiling problem from statistical mechanics and chaotic mapping

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We introduce a *partial decidability protocol* for the Wang tiling problem (which is the prototype of undecidable problems in combinatorics and statistical physics) by constructing a suitable mapping from tilings of finite squares of different sizes. Such mapping depends on the initial family of Wang tiles (the "alphabet") with which one would like to tile the plane. This allows to define effective entropy and temperature associated to the alphabet (together with the corresponding partition function). We identify a subclass of *good alphabets* by observing that when the entropy and temperature of a given alphabet are "well-behaved" in the thermodynamical sense then such alphabet can tile the infinite two-dimensional plane. Our proposal is tested successfully with the known available good alphabets (which produce periodic tilings, aperiodic but self-similar tilings as well as tilings which are neither periodic nor self-similar). Our analysis shows that the Kendall's Tau coefficient is able to distinguish alphabets with a good thermodynamical behavior from alphabets with bad thermodynamical behavior. The transition from good to undecidable behavior is related to a transition from non-chaotic to chaotic regime in discrete dynamical systems of logistic type.

Contents

I. INTRODUCTION

The Wang tiling problem [1] is one of the deepest and most relevant problem in statistical physics, mathematics and computer science. A Wang tile is a square with one color for each side. Given a finite family ("alphabet") of Wang tiles (from now on, a Wang tile will simply be denoted as tile), the problem is to understand whether or not one can cover the plane using these tiles[1] and satisfying certain rules. Here it is the original formulation [1]:

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[1] The rule of the game is that one can use translated copies of these tiles in such a way that two tiles can be put together along one side if and only if the common side of two neighbor tiles has the same color in both. Moreover, one can only use the tiles of the initial family but one cannot rotate the tiles.

Partial decidability Protocol for the Wang tiling Problem from Statistical Mechanics and Chaotic Mapping

→ The Green part of the title Justifies why this can be a physics talk!!

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Organization of the talk

*) What are undecidable problems?

*) Examples of undecidable problems in physics

*) The Wang tiling problem (WTP)

*) Statistical Mechanics approach to WTP

*) Partial decidability criteria from natural physical requirement

*) Positive temperature and/or non-Chaotic mapping

*) Undecidability and Transition to Chaos

*) Conclusions and perspectives

"Assume we are given a finite set of square plates of the same size with edges colored, each in a different manner. Suppose further there are infinitely many copies of each plate (plate type). We are not permitted to rotate or reflect a plate. The question is to find an effective procedure by which we can decide, for each given finite set of plates, whether we can cover up the whole plane (or, equivalently, an infinite quadrant thereof) with copies of the plates subject to the restriction that adjoining edges must have the same color."

This problem hides some of the most important open questions both in mathematics and physics. As Wang himself noticed, the existence of aperiodic tile sets is closely related to this decision problem. In particular, if aperiodic tile sets would not exist, then the Wang tile problem would be decidable. In fact, as it was shown by Berger [2] (which was Wang student) aperiodic tile sets do indeed exist and this fact implies the undecidability of the problem.

A small remark is in order: the sentence "to solve the Wang tiling problem" actually means "to find an algorithm that, when receiving the input (the initial family of tiles, the alphabet) produces the corresponding output (which is **yes** if the alphabet can cover up the whole plane and is **no** if it cannot)". A problem of this type is called *decision problem* (since the answer is either yes or not). A decision problem is called undecidable if no algorithm can solve it in *all instances*. In the present case, the undecidability of the Wang tiling means that *there is no algorithm able to solve the problem for any alphabet*. As it will be explained in details in the following sections, the main idea of the present manuscript is to define (*in the space of all alphabets*) a suitable boundary separating the alphabets for which the decision problem can be solved from the ones for which it cannot. Such boundary is closely related to the transition to chaos in discrete dynamical systems.

For the sake of clarity, it is important to remark here the following fact. We will propose a criterion, inspired by statistical mechanics, which allows to determine a subset of alphabets which tile the plane. On the other hand, when a given alphabet does not satisfy such criterion (to be introduced in the next section), it does not necessarily mean that the alphabet does not tile the plane (otherwise the problem would be decidable). This fact only means that the behavior of this alphabet is (in a precise sense to be specified in the next sections) chaotic when one increases the size of the square to tile (something which prevents one from guessing what happens in the case of very large squares). Thus, when the alphabet of interest does not satisfy our criterion, we can only conclude that there is no efficient algorithm to decide whether or not the given alphabet tiles the plane (thus, the alphabet belongs to the genuine undecidable region of the alphabets space).

Let's go back to the Wang tilings. Berger results were later improved by Robinson [3] which constructed a smaller aperiodic tile sets. A very important work has been the one of Jeandel and Rao [4] who showed that the smallest aperiodic tileset must have eleven tiles (also providing examples). A very powerful structure appearing in most of the known examples of aperiodic tile sets is a hierarchical self-similar structure (notable examples are [4] and [5]). However, there is one important exception: the tiling discovered in [6] which is neither periodic nor self-similar (unlike most of the known aperiodic tilings).

The goal of the present paper is to define suitable temperature, entropy and partition function associated to any alphabet Γ in order to identify a subset of alphabets which are able to cover up the whole plane. Thus, in order to achieve this partial decidability protocol (of an otherwise undecidable problem), statistical mechanics plays a fundamental role.

The Wang tiling problem has further direct connections with statistical physics. The aperiodic structures (which, on the mathematical side, are at the heart of the undecidability of the Wang tiling problem) have been discovered experimentally in [7] [8]. Interesting statistical mechanics models (where the tiles represent clusters of atoms while the kinematical constraints which these atoms must satisfy are represented as forbidden matching between the tiles) took the Wang tiles as inspiration [9]. In these statistical models the above mentioned aperiodic structures play a fundamental role. For instance, in [10] (in a model with short range interactions), it was discussed the appearance (in the self-similar structure characteristic of Wang tiles) of an ordered phase with periodic behaviors at increasingly larger scales as the temperature decreases. Closely related examples of models with chaotic temperature dependence near zero (when the Gibbs measure may diverge with, correspondingly, volatile variations in the macroscopic state of the systems under investigation) have been constructed in [11] [12] (for a very nice pedagogical review of these models see [13] and references therein).

The relations of the Wang tiling problem with physics are even deeper. These can be seen in the analysis of undecidability in mathematics and physics (which started soon after Godel results: see [14] and references therein). In recent years, we have observed a huge burst of interest in the appearance of undecidability in fundamental physical problems. Two results have influenced a lot the present work: the undecidability of the spectral gap [15] and the undecidability of thermalization in quantum many-body [16]. As far as the undecidability of the spectral gap, it has been shown that, for 2D quantum many-body systems with nearest neighbor and translational invariant interactions that the spectral gap is undecidable. As far as the problem in thermalization is concerned, it has been shown that the central issue to understand whether or not a given system thermalizes is undecidable as well.

In the present manuscript we will exploit the relations between Wang tilings and the appearance of chaotic structures. However, the present approach (as it will be discussed in details in the next sections) is completely different

What are undecidable problems?

Undecidable problems are at the heart of the most important building blocks of Mathematics, physics, computer science ...

In a Rough form, some of these were already known in ancient Greek times

*) "this sentence is a lie"

**) "Which is the smallest integer that in order to be defined needs more than 150 letters?"

→ these two paradoxes show that undecidable Statements appear not only due to self-References But also in Relation with minimality issues

The Brightest Examples are K. Gödel theorems (1931)

Very Roughly speaking, a consequence of these is that if Mathematics is consistent then there are true statements which cannot be proven with the Axioms and inference rules of Mathematics

Then, in 1936, A. Turing in his foundational paper on Turing machine showed that the Halting Problem is also undecidable

→ **Halting Problem:** find program H (Python, C++, ...) such that H can decide whether or not another (generic) program P on a given Input I stops or Run forever.

Turing showed that such problem is undecidable

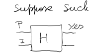
Namely, there is no such H that can work on any (P, I)

Note that Both the Program and the Input can be mapped into Binary Strings of 0s and 1s

Although we know very well that Turing result is very concrete since it asserts the impossibility of a universal "debugging Machine" (the Dream of any computer scientist) at that time (1936) computers did not exist

Thus, Both Godel results and Turing results were considered "Twisted" "unnatural" In the sense that many researchers believed that Mathematical problems which appear in usual applications in Science do not belong to the "undecidable" class

→ **Turing Proof:** suppose such H actually exists:

*) 

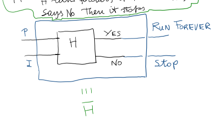
means P stops on I

*) 

means that P enters in an infinite loop on I and never stops

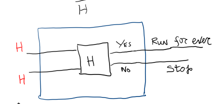
Then, if H exist, I can also construct $\bar{H}$ ~ "Not − H"

$\bar{H}$: if H on (P, I) says Yes, then $\bar{H}$ runs forever, if H on (P, I) says No then it stops



But then, we can give as Inputs (P, I) to $\bar{H}$ the couple (H, H)

$\bar{H}$



if we now give (H, H) as Input to $\bar{H}$ there are 2 possibilities

1) if H given to H stops then it does not stop

2) if H given to H does not stop then it stops

} → We get a contradiction

from the references described here above. The typical chaotic behavior which appears in those references is closely related to the presence of aperiodic tilings. On the other hand, in the present manuscript, we will associate a suitable discrete mapping to any family of Wang tiles (namely, to any *Wang alphabet*). When such mapping is not chaotic for a given alphabet, then such an alphabet (which will be denoted as *good alphabet*) can cover up the whole plane. On the other hand, when such mapping is chaotic, we cannot decide about the corresponding alphabet. In other words, we will define a *partial decidability protocol for Wang tilings.*

The connection between the present approach to the Wang tiling problem and the physics of chaotic systems (for a review see [17]) has its roots in the linkage between deterministic chaos and undecidability which is already well known since, at least, Chaitin's theorems [18]. It is, by now, pretty clear that determinism and undecidability can coexist and, therefore, randomness and determinism can live together as well (see, for instance, the detailed analysis in [19] [20] [21]). Very roughly speaking, undecidability can manifest itself in dynamical systems with chaotic behavior (in the present case, we will be interested in discrete dynamical systems). In some cases, the existence of deterministic chaos in dynamical systems can be related to a sort of Godelian undecidability rather than to the typical numerical untractability (so characteristic of chaotic systems). Quite fittingly, the discrete mapping that will be defined in the following sections to identify good alphabets, manifests a clear chaotic tendency when the alphabet is not good. Thus, the transition between good and bad alphabets (within the space of all alphabets) looks similar to a transition from regular to chaotic behavior in a discrete dynamical system.

From the computer science viewpoint, the present results are also very useful. Indeed, the problem of determining whether a given finite set of square tiles can tile a simply connected region of the plane is known to be NP-complete, a complexity classification that implies no known polynomial-time algorithm exists to solve it in the general case [28]. This inherent computational hardness means that exhaustive search methods, which require exponential time and resources, are often infeasible for practical purposes. In particular, [26] demonstrated that tiling simply connected regions with a fixed set of rectangles (including squares as a special case) is NP-complete, even when the number of tile types is large but finite. These results establish the fundamental intractability of the problem and highlight the necessity of developing efficient criteria or "recipes" to decide "tilability" without resorting to exhaustive search. Given this NP-completeness, it becomes crucial to develop criteria or "recipes" that can efficiently decide "tilability" without resorting to brute-force search, which is prohibitive in terms of both memory and energy consumption. Such recipes aim to identify polynomial-time verifiable conditions or invariants that guarantee "tilability" or "non-tilability", thereby circumventing the exponential complexity of the general problem. This approach aligns with recent advances exemplified by [27], who provided polynomial-time algorithms for special cases of tiling Polyominoes with fixed-size square tiles and packing dominoes, leveraging compact geometric representations of the regions. It is worth emphasizing that our approach can be easily extended to different types of tiling problems (even in higher dimensions).

Our work builds upon these foundations by proposing a generalized framework that defines combinatorial and geometric invariants serving as certificates of "tilability" for tiles alphabets. By verifying these invariants, one can preemptively determine the feasibility of tiling a region, thus avoiding unnecessary computational overhead. This not only contributes to the theoretical understanding of "tilability" within the P versus NP landscape but also offers practical benefits in applications where resource constraints are critical.

This paper is organized as follows: in the second section we introduce the discrete mapping associated to the Wang tiling problem together with the "decidability" protocols. In the third section we discuss how to separate good from bad alphabets. In the fourth section, we will discuss the relations of the present approach with discrete chaos. In the fifth section, we will discuss how to quantify "goodness" and "badness" of alphabets. In the sixth section, we discuss what we have done and, especially, what we have not done. In the final section some conclusions will be drawn.

II. THE WANG TILING PROBLEMS AND THE $W_\Gamma(n)$ MAPPING

As it has been already emphasized in the introduction, a Wang tiles set Γ or "alphabet" is a collections of a fixed number of squares (the number of square tiles of the alphabet will be denoted as the size of the alphabet $q = |\Gamma|$). Each edge of the square possesses a color (a discrete label). The matching rules are the following:

1) The squares can be neither rotated nor reflected

2) Two square can be matched side to side only if the right edge of the left square has the same color as the left edge of the right square.

3) Two square can be matched top to bottom only if the bottom edge of the square on the top has the same color as the top edge of the square on the bottom.

Then, we want to know if the given alphabet Γ of size q can cover up the whole plane. This question has been shown to be undecidable *in the general case* [2]. However, it is extremely interesting to understand how and when undecidability sets in in practice. The idea is to find a concrete boundary in the space of all alphabets Γ separating

Undecidable problems in Math / Physics

It is now clear that undecidable problems are actually quite common

*) Spectral Gap in several Spin Systems is undecidable!

*) The Existence of a SUSY ground state in several Supersymmetric theory is also undecidable

*) Diophantine equations

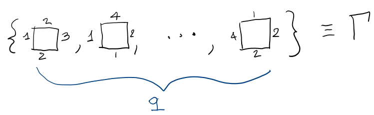
⋮

Thus, the question of "undecidability" in physics should be taken more seriously

Wang Tiling problem

I will discuss a very deep undecidable problem in Mathematics with many relations with physics (in particular, with aperiodic Tilings)

Consider a set of q different square tiles with a color on each side:

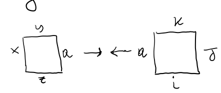


We say that the size of the family Γ is q:

$$|\Gamma| = q$$

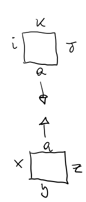
The Wang tiling problem is the following: given a family Γ with $|\Gamma| = q$, can we tile with Γ the whole $\mathbb{R}^2$ without holes with the following rules?

Rule 1:



I can match side-to-side if and only if the Right edge of the left tile has the same color as the left edge of the Right tile

Rule 2:



We can match top-to-Bottom if and only if the Bottom edge of the top tile has the same color as the top edge of the Bottom tile

Rule 3: the tiles cannot be rotated

Wang: If all the tiling of $\mathbb{R}^2$ are periodic then the problem is decidable

Berger (Wang's student) Constructed an example of an a-periodic tile $\Longrightarrow$

The Wang tiling problem is undecidable

the decidable instances from the undecidable ones. The results of the present manuscript strongly suggest that in the space of all alphabets such a boundary can be defined and, moreover, share many features with the transition to chaos in discrete dynamical systems and can be identified using sound arguments from statistical mechanics.

In order to proceed, let us define the following quantity:

$$W_\Gamma(n) \overset{def}{=} \{number\ of\ different\ tilings\ of\ a\ n \times n\ squares\ with\ the\ alphabet\ \Gamma\} \ . \tag{1}$$

Let us also define *the energy* $E_\Gamma(n)$ *of an alphabet* Γ on a $n \times n$ *square* as the area $A(n)$ of the square (namely n^2):

$$E_\Gamma(n) \overset{def}{=} A(n) = n^2 \ . \tag{2}$$

Thus, from the above definitions it is clear that $W_\Gamma(n)$ *plays the role of the degeneracy of the energy level* $E_\Gamma(n)$. Therefore, we can define *the entropy* $S_\Gamma(n)$ of Γ *at the energy level* n as:

$$S_\Gamma(n) \overset{def}{=} \log W_\Gamma(n) \ . \tag{3}$$

In the following it will be convenient to use the $\log_{10}$ instead of the usual ln, but conceptually there is no difference. These definitions are natural for the following reasons.

On the one hand, we expect that for alphabets Γ with good thermodynamical behavior there will be many different ways to tile a $n \times n$ square (since for most of physically reasonable systems the degeneracy of the energy level is an increasing function of the energy). This, in particular, implies that $W_\Gamma(n)$ is different from zero for arbitrarily large n. This last conclusion (which is a rather obvious fact in many physical systems) precisely means that Γ can tile arbitrarily large regions! Hence, for such *good alphabets* (namely, alphabets with a good thermodynamical behavior), both $W_\Gamma(n)$ and $S_\Gamma(n)$ should be increasing function of n and, consequently, good alphabets can tile arbitrarily large regions. This last observation is equivalent to the typical thermodynamical condition related to the positiveness of the temperature:

$$\frac{\partial S}{\partial E} > 0 \Leftrightarrow \frac{\partial S_\Gamma(n)}{\partial n} > 0 \ . \tag{4}$$

Conversely, from the intuitive viewpoint one can assume that an alphabet Γ with a good thermodynamical limit[2] should have a well defined positive temperature and, consequently, Eq. (4) should be satisfied.

A very important function which captures the main thermodynamical features of a system is its partition function. Using the previous definitions we can define *the partition function of the alphabet* Γ *as* Z_Γ:

$$Z_\Gamma(\beta) \overset{def}{=} \sum_n W_\Gamma(n) \exp\left(-\beta n^2\right) \ . \tag{5}$$

The above partition function encodes relevant informations about the alphabet Γ. For instance, if the alphabet Γ does not tile the plane then there exist an n^* with the property that

$$\forall n > n^* \ , \quad W_\Gamma(n) = 0$$

(as otherwise, Γ would actually tile the plane[3]). In this case, the partition function is analytic as $Z_\Gamma(\beta)$ will be just a polynomial in $\exp(-\beta)$. On the other hand, if $Z_\Gamma(\beta)$ is not analytic as function of β then necessarily there are infinitely many terms in the sum on the right hand side of Eq. (5) (otherwise the function would be analytic) and this means that actually Γ tiles the whole plane. Consequently, the properties of $Z_\Gamma(\beta)$ are also worth to be further investigated (we will come back on them in a future publication), but here below we will discuss a more direct approach to disclose the arising of undecidability.

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[2] Namely, an alphabet which can tile aribitrarily large squares and which, therefore, allows to define the thermodynamical limit with the above definitions of entropy and energy in Eqs. (2) and (3).

[3] Indeed, the negation of the formula "$\forall n > n^* \ , \ W_\Gamma(n) = 0$" is precisely that $W_\Gamma(n)$ is different from zero for arbitrarily large n and so Γ tiles the plane.

After Berger,

A very interesting and deeply analyzed issue is the following:

Construct aperiodic tiles with the smallest possible 9

It is known that the smallest possible 9 leading to aperiodic tiles is 9=11

Untie now, almost all the known examples of aperiodic tiling are self-similar (Kind of Sierpinsky structure)

there is only one example wich is neither periodic nor self-similar

Decidability Protocol ?

the Wang tiling problem is connected with many relevant problems in combinatorics, computer science and physics.

the Wang tiling problem is NP-complete: This means that if we understand more WTP we understand more of all the NP complete problems

It would be fantastic to have a protocol (an algorithm) such that, given a family Γ, we can know whether or not Γ tiles the plane. This of course is impossible due to the fact that the Wang tiling problem is undecidable. But should we give up entirely?

A. Protocol I

In order to devise a partial "decidability" protocol able to detect a subclass of good alphabet, the requirement to satisfy the condition in Eq. (4) is a good starting point. Obviously, from the practical viewpoint, one cannot compute $W_\Gamma(n)$ for arbitrarily large n due to the fact that this would need an infinitely powerful computer (which is not available). Thus, in order to use the present ideas based on statistical mechanics in concrete situations, one can only verify that the condition in Eq. (4) is satisfied for finite n. Thus, the question is: how large should the set (1, 2, 3,.., n_{max}) be in order to be confident that the numerical verification of Eq. (4) for (1, 2, 3,.., n_{max}) ensures "tilability for arbitrarily large n"? We do not have rigorous results on this point. However, the numerical results to be described in the following sections strongly suggest that it could be enough to verify Eq. (4) for all $n \leq n_{max}$ with

$$n_{max} \gg |\Gamma| = q. \tag{6}$$

In the examples here below it has been enough to take n_{max} 2 or 3 times q. This strategy is very simple to implement.

B. Protocol II

On the other hand, it is possible to define a second criterion to detect the "goodness" of a given alphabet Γ which is also able to disclose a very intriguing relation with the transition to chaos in discrete dynamical systems. The main idea is to plot $W_\Gamma(n+1)$ in terms of $W_\Gamma(n)$. In this way, the mapping f_Γ is computed numerically plotting on the vertical $Y-$axis $W_\Gamma(n+1)$ and on the horizontal $X-$axis $W_\Gamma(n)$. In this plots, the two coordinates of each point P are:

$$P = (X, Y) = (W_\Gamma(n), W_\Gamma(n+1)) \ . \tag{7}$$

The mapping obtained in this way defines (at least locally) $W_\Gamma(n+1)$ as a function F_Γ of $W_\Gamma(n)$:

$$W_\Gamma(n+1) = F_\Gamma(W_\Gamma(n)) \ . \tag{8}$$

Such function F_Γ is the sought discrete mapping. We will do the same with the entropy, plotting $S_\Gamma(n+1)$ in terms of $S_\Gamma(n)$ obtaining

$$S_\Gamma(n+1) = f_\Gamma(S_\Gamma(n)) \ , \tag{9}$$

$S_\Gamma(n+1)$ as a function f_Γ of $S_\Gamma(n)$. Due to the undecidable nature of this combinatorial problem, it may be very difficult to be able to compute explicitly such function f_Γ for any alphabet Γ. However, as it will be now discussed, this is not really necessary. The present analysis shows that f_Γ behaves very differently depending on whether or not there is an "easy" algorithm to decide that Γ can cover the whole plane. These two very different behaviors allows to deduce many non-trivial results even if the analytic form of f_Γ is not known precisely.

III. SEPARATING "GOOD" FROM "BAD" MAPPINGS

To properly assess and classify the different alphabets as either "good" or "bad", we analyze for each case three key visualizations: the geometric structure of the tiling, the complexity growth plot $W_\Gamma(n)$ versus n, and the phase-space plot $W_\Gamma(n+1)$ versus $W_\Gamma(n)$. These plots jointly offer a comprehensive perspective on the spatial and combinatorial dynamics associated with each alphabet. For each case, a single figure presents the three components together, enabling a direct visual comparison of their respective behaviors. In the discussion that follows, we adopt the notation G_i to denote alphabets that are considered "Good" either due to established results in the literature or based on favorable tiling behavior observed through simulation. Conversely, alphabets denoted as B_i are classified as "Bad", reflecting either their known limitations as reported in the literature or pathological behaviors detected in computational experiments. This classification guides our comparative analysis and is supported by the visual and quantitative characteristics displayed on each plots. Further details will include presenting similar plots for the associated entropy $S_\Gamma(n)$, which will offer additional insights into the alphabet's thermodynamic behavior.

Alphabet G1. This is a straightforward minimal example of a "good" alphabet. It consists of only two tiles, yet it is capable of completely mapping the two-dimensional plane. The plot of $W_\Gamma(n)$ depicts a simple alphabet exhibiting monotonic growth in all directions. Consequently, $W_\Gamma(n)$ increases with structure size n , and the corresponding $W_\Gamma(n+1)$ vs. $W_\Gamma(n)$ plot exhibit linear growth , reflecting the inherent simplicity and full coverage of the alphabet. Illustrations can be seen on **FIG.** (2)

A physical approach

Due to the importance of the Wang tiling problem, even partial results would be great! Can we know, at least in some cases, whether or not a given family Γ tiles $\mathbb{R}^2$?

In other words, we are searching for an Heuristic/Ansatz

→ Our goal: to find a protocol / a program / a criterion that, at least, is able to identify the good families found so far in the literature (no such criterion has been proposed so far): can we do better?

Help from Statistical Mechanics

Let us define our statistical system as follows:

Given a family Γ such that $|\Gamma| = q$, we define as the energy of the system on a $n \times n$ square as n^2:

$$E_\Gamma(n) = n^2,$$

while the corresponding degeneracy is

$$W_\Gamma(n) \overset{\text{def}}{=} \left\{ \begin{array}{l} \text{number of different tilings of} \\ \text{a } n \times n \text{ square with the family } \Gamma \end{array} \right\}$$

Thus, we can associate the following partition function to the family Γ:

$$Z_\Gamma(\beta) = \sum_n W_\Gamma(n) e^{-\beta n^2}$$

Physical considerations

1) Since $W_\Gamma(n)$ is the degeneracy of the n-th energy level, we can define the following entropy:

$$S_\Gamma(n) = \log(W_\Gamma(n))$$

2) Suppose that the family Γ does not tile the plane, then $\exists\, n^* \mid W_\Gamma(n) = 0 \ \forall n > n^*$

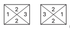
In this case the partition function is analytic in β as it is just a polynomial (there are finitely many terms).

3) Viceversa, suppose that Z_Γ is not an analytic function of β, then there are necessarily infinitely many terms in Z_Γ → otherwise it would be analytic

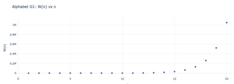
and so, $\forall n, \exists\, n^* > n \mid W_\Gamma(n^*) \neq 0$

But this means that Γ tiles the plane

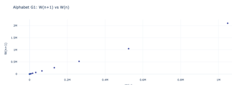
FIG. 1: *



(a) Alphabet G1



(b) $W_\Gamma(n)$ vs n



(c) $W_\Gamma(n+1)$ vs $W_\Gamma(n)$

FIG. 2: Alphabet G1 and its associated complexity plots.

Alphabet G2. Cited from [25], this alphabet is an example of an aperiodic, self-similar tiling system. The behavior of $W_\Gamma(n)$ is nonlinear but exhibits a recognizable monotone behavior. The $W_\Gamma(n+1)$ vs. $W_\Gamma(n)$ plot reveals the recursive and hierarchical construction of the tiling. This self-similarity is a distinguishing feature that supports its classification as a "good" alphabet. Illustrations are provided in **FIG.** 4.

Alphabet G3. Described in [4], this alphabet represents an aperiodic but non-self-similar system. Unlike G2, the lack of self-similarity manifests in more irregularities within the $W_\Gamma(n)$ growth, and the $W_\Gamma(n+1)$ vs. $W_\Gamma(n)$ plot does not present a fractal pattern. However, the absence of periodicity and the tiling completeness render it a member of the "good" class, albeit with more complex dynamical behavior. Illustrations can be seen on **FIG.** (6)

Alphabet B1. This simulated "bad" alphabet demonstrates irregular growth in $W_\Gamma(n)$ and exhibits discontinuous transitions in the $W_\Gamma(n+1)$ vs. $W_\Gamma(n)$ plot. The lack of structure in both plots reflects the limitations of the alphabet to generate a coherent or comprehensive coverage of the plane. Illustrations can be seen on **FIG.** (8)

Alphabet B2. Also generated via simulation, this alphabet fails to produce a stable or predictable $W_\Gamma(n)$ trajectory. The scatter in the $W_\Gamma(n+1)$ vs. $W_\Gamma(n)$ plot further indicates poor local consistency and an absence of long-range ordering, reinforcing its categorization as "bad" as the one showed by alphabet B1. Illustrations can be seen on **FIG.** (10)

Alphabet B3. Introduced in [4] as a counterexample to prevailing trends, this alphabet displays anomalous behavior in both plots. Although its $W_\Gamma(n)$ initial growth superficially resembles that of some well-behaved alphabets, its $W_\Gamma(n+1)$ vs. $W_\Gamma(n)$ plot uniquely reveals a stagnant growth that leads to structural inconsistencies. This significant deviation from expected patterns suggests a transitional or deceptive character, challenging straightforward

Advantages of the physical approach

It is clear that the above physical definitions encode all the relevant properties of a family Γ.

Even more is true!

Partial decidability protocols

Protocol 1

Physics intuition gives a useful hint.

First of all, as already emphasized, if our Z_Γ possesses a thermodynamic limit, then Γ tiles the plane.

Moreover, for most physical systems with a thermodynamical limit, the entropy is an increasing function of the energy: $\frac{\partial S}{\partial E} > 0$

In our case, the above condition reduces to:

$$\boxed{\frac{\partial W_\Gamma}{\partial n} > 0} \quad (*)$$

This simple criterion (equivalent to requiring that the temperature is positive) gives rise to the

Protocol I: if the condition (*) is verified for your family Γ of interest then there is a good possibility that your family Γ tiles the plane!

Quite Remarkably this protocol ("Euristic") is able to identify all the families of the literature

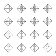
Limitations of the Method

1) Note that this is only "Euristic":
Namely, in principle there can be Γ satisfying the criterion and which, nevertheless, do not tile the plane

2) Due to the limitations of the Hardware, the condition $\frac{\partial W_\Gamma}{\partial n} > 0$ can be verified only for a finite number of steps N^*. It would be ideal to have $N^* \gg 1$

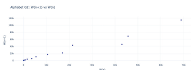
However, quite surprisingly, in all the cases we checked successfully $N^* \sim 29$ was enough

FIG. 3: *



(a) Alphabet G2

(b) $W_\Gamma(n)$ vs n



(c) $W_\Gamma(n+1)$ vs $W_\Gamma(n)$

FIG. 4: Alphabet G2 and its associated complexity plots.

classification. Illustrations can be seen in **FIG.** (12).

Through this comparative study, it becomes clear that the distinction between "good" and "bad" alphabets is more nuanced than a binary classification. The behavior of W_Γ under sequential growth, as visualized in the two standard plots, offers a powerful lens to probe underlying structural properties and dynamic complexity of tiling alphabets.

To effectively compare alphabets with vastly different $W_\Gamma(n)$ values, it is crucial to address the scale issue. Directly analyzing $W_\Gamma(n)$ can obscure underlying patterns, as its magnitude can vary widely across different alphabets. Therefore, we propose utilizing the entropy $S_\Gamma(n)$ as the primary variable for analysis, defined as:

$$S_\Gamma(n) = \log_{10}(W(n)) \tag{10}$$

This logarithmic transformation effectively normalizes the scale of $W_\Gamma(n)$ across diverse alphabets, thereby enabling a more meaningful and robust comparison, particularly within the context of regression tasks. Such normalization is essential for achieving consistent and interpretable analytical results. The significant impact of this transformation

Protocol II

The second protocol provides with more info not only on the positives but also on the negatives

Chaos and 1-Dimensional Mapping

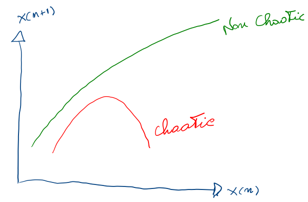
let us consider the following 1-D discrete differential equation

Logistic Equation

$$x(n+1) = -r\, x(n)(1 - x(n))$$

depending on the value of the parameter r the above equation can have regular or chaotic behavior, in general, if one considers

$$x(n+1) = f(x(n))$$



depending on the shape of the function $f(x)$ the discrete differential equation can be chaotic or not

Appearance of Chaos

If the function $f(x)$ has a local "peaked" maximum then the discrete ODE is likely to be chaotic

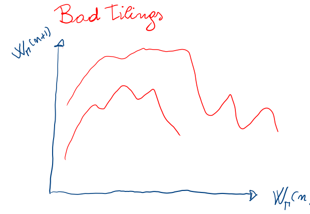
Big Idea: Regular (non-Chaotic) mapping are good!

Instead of plotting $W_\Gamma(n)$ vs n or $S_\Gamma(n)$ vs n

Let us plot $W_\Gamma(n+1)$ vs $W_\Gamma(n)$ or $S_\Gamma(n+1)$ vs $S_\Gamma(n)$

Good Tilings

Bad Tilings



Thus we clearly see that Bad tilings have manifest tendency to be chaotic
Good tilings on the other hand do not have local maxima!

Using both of these protocols we have been able to identify successfully all the good families present in the literature.

Our protocols are the only known a priori test available in the literature which have been able to identify all the good families
(periodic, non-periodic but self-similar, non-periodic non-self-similar)

Potential applications

The potential applications are huge:
The Wang-tiling problem belongs to the NP-complete class
This means that many other famous problems in Stat. Mech., Computer Science, discrete Math. can be mapped into Wang Tiling Problems
Thus, we can use our approach as criteria to guess a priori whether or not a given problem has a solution!

Limitations

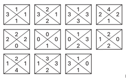
*) There is a practical limitation due to the fact that we can test our criteria only for a finite range of square sizes.
Namely, we can arrive until a square $N_{Max} \times N_{Max}$
We would like $N_{Max} \gg 1$

But we can only arrive at $N_{Max} \sim 31$ at best

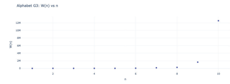
*) Rigorous sufficient conditions are hard to establish

Despite this, our protocols work anyway!

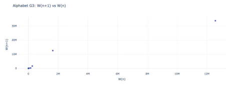
FIG. 5: *



(a) Alphabet G3



(b) $W_\Gamma(n)$ vs n



(c) $W_\Gamma(n+1)$ vs $W_\Gamma(n)$

FIG. 6: Alphabet G3 and its associated complexity plots.

on regression analysis is clearly demonstrated in **Table I**, which presents the derived regression coefficients using $S_\Gamma(n+1)$ vs $S_\Gamma(n)$.

To rigorously analyze the growth monotonicity of $S_\Gamma(n+1)$ versus $S_\Gamma(n)$ as defined in Eq.(11), we perform a regression analysis for each alphabet. Instead of Pearson's correlation coefficient, we employ Kendall's Tau (τ) to assess the relationship between data pairs. This choice is specifically motivated by our primary focus on monotonicity rather than strict linearity, and by the superior robustness of Kendall's Tau to outliers. Subsequently, we compare the alphabets based on their respective regression coefficients c_0 and γ, as well as the Kendall's Tau coefficient (τ) , as summarized in **FIG.** (14 to 24) and **Table** (I).

For the good alphabets (namely, the alphabets which can cover the whole plane) $f_\Gamma\left(S_\Gamma\left(n\right)\right) = f_\Gamma\left(z\right)$ is very close to a power low (with very small oscillations):

$$f_\Gamma\left(z\right) = c_0 z^\gamma \left(1 + \omega(z)\right) \ , \quad \gamma > 0 \ , \tag{11}$$

Conclusions

*) Using sound arguments from statistical mechanics and from the theory of Chaos we have found the first effective a priori tests that are able to identify the family of Wang tiles which can tile $\mathbb{R}^2$

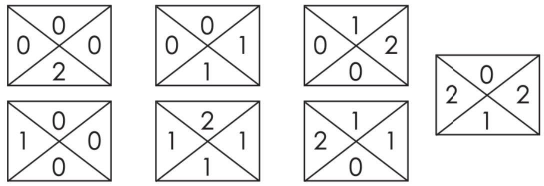
*) Our approaches also disclose a close relation between the "Transition from decidable to undecidable" and the Transition from "regular to Chaotic behavior" in discrete dynamical system → negative criteria → When your family Γ of interest generates a mapping which is chaotic then it is unlikely that one can prove that Γ tiles the $\mathbb{R}^2$ even if it does

*) We can apply our strategy to other NP-complete / undecidable problems
(Graph Theory) (Spectral Gap ...)

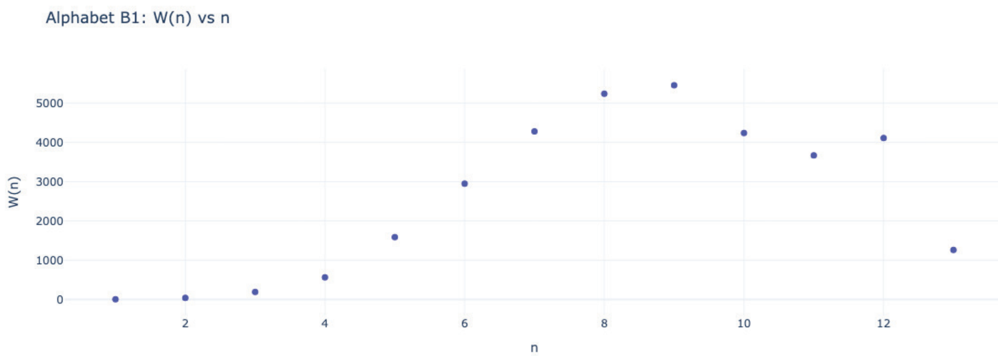
*) In particular: Spectral Gap problem and transition to Chaos ?

¡Thank You Very Much!

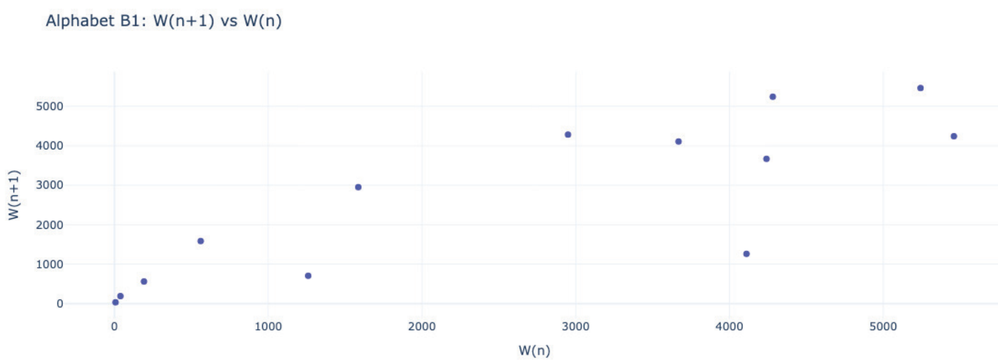
FIG. 7: *



(a) Alphabet B1



(b) $W_\Gamma(n)$ vs n



(c) $W_\Gamma(n+1)$ vs $W_\Gamma(n)$

FIG. 8: Alphabet B1 and its associated complexity plots.

where c_0 is a constant and $\omega(z)$ is an oscillating function with a small amplitude:

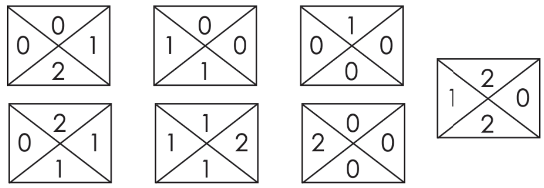
$$|\omega(z)| \ll 1 \ . \tag{12}$$

Alphabet	c_0	γ	τ
G1	1.25	0.899	1.00
G2	1.49	0.754	1.00
G3	1.54	0.810	1.00
B1	1.75	0.550	0.72
B2	1.75	0.550	0.72
B3	1.86	0.596	0.79

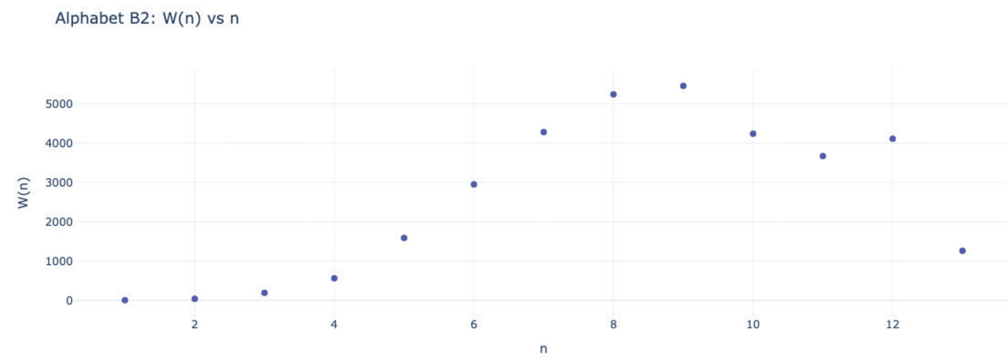
TABLE I: Table I: Alphabets Regression coeff

Figure 2 presents the complexity analysis for Alphabet G1. This alphabet demonstrates a remarkably predictable growth pattern, characteristic of "good" alphabets, evident in both its entropy plots and regression coefficients. Figure

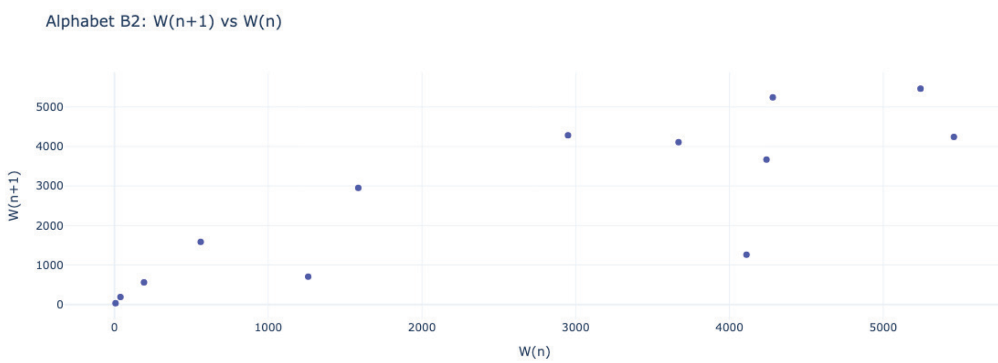
FIG. 9: *



(a) Alphabet B2



(b) $W_\Gamma(n)$ vs n



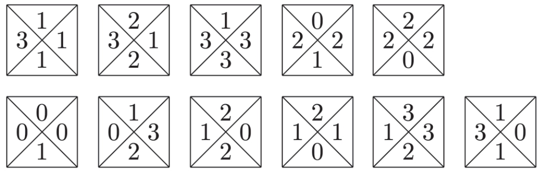
(c) $W_\Gamma(n+1)$ vs $W_\Gamma(n)$

FIG. 10: Alphabet B2 and its associated complexity plots.

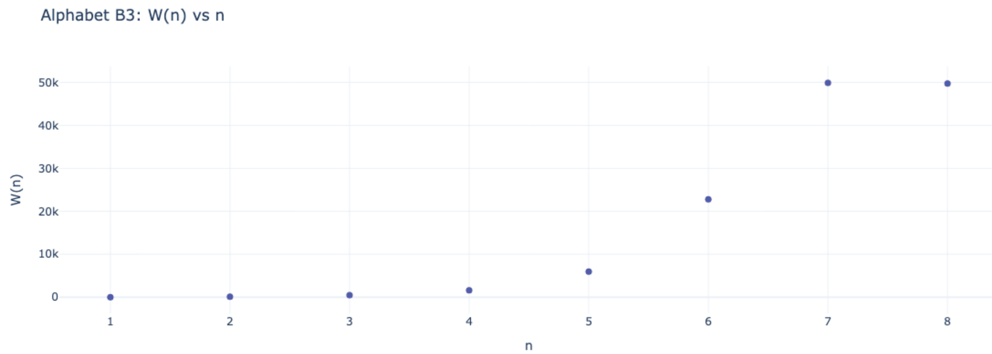
14(b) shows a consistent and monotonic increase in $S_\Gamma(n)$ with n, indicating a stable, non-erratic growth in the number of distinct tiles. The core analysis of growth monotonicity, as depicted in Figure 14(c) plotting $S_\Gamma(n+1)$ against $S_\Gamma(n)$, reveals an almost perfectly linear relationship between successive entropy values. This strong linearity is quantitatively supported by the regression coefficients from Table I: a scaling factor $c_0 = 1.25$ indicates a consistent base growth rate, and an exponent $\gamma = 0.899$ (approaching unity) signifies a nearly linear relationship in the logarithmic domain, consistent with stable complexity evolution. Crucially, a Kendall's Tau coefficient of $\tau = 1.00$ confirms a perfect positive monotonic relationship, meaning $S_\Gamma(n+1)$ invariably increases with $S_\Gamma(n)$, validating the consistent and predictable growth inherent to "good" alphabets. This empirical evidence for minimal oscillation strongly supports the theoretical condition $|\omega(z)| \ll 1$ in Equation (11).

Figure 16 displays another example of a "good" alphabet (G2). It is illustrated the growth of $S_\Gamma(n)$ with n, showing a consistent, albeit slightly less uniform, monotonic increase in entropy compared to G1. This indicates a generally stable increase in tiles complexity. The relationship between $S_\Gamma(n+1)$ and $S_\Gamma(n)$, depicted in Figure 16(c), reveals a positive linear correlation, with most data points closely adhering to the regression line. From Table I, Alphabet G2 exhibits regression coefficients $c_0 = 1.49$ and $\gamma = 0.754$. While γ is slightly lower than G1, it still signifies a robust, near-linear growth in the logarithmic domain, consistent with predictable complexity. More critically, the Kendall's Tau coefficient $\tau = 1.00$ confirms a perfect positive monotonic relationship, reinforcing that $S_\Gamma(n+1)$ consistently increases with $S_\Gamma(n)$. This perfect monotonicity, combined with the high linearity, categorizes Alphabet G2 as having

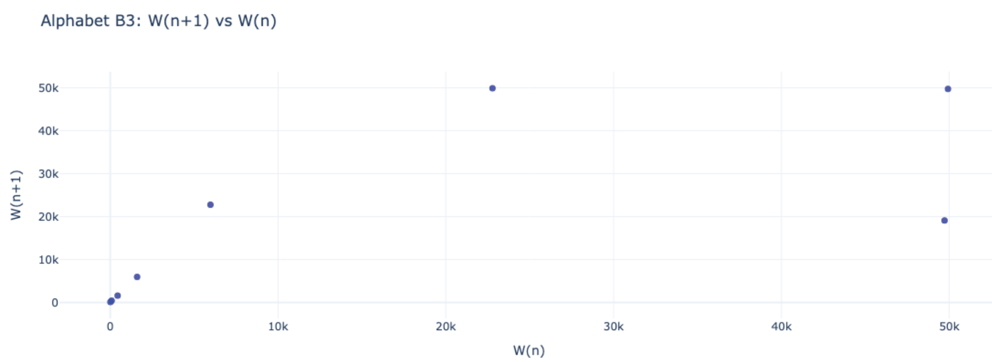
FIG. 11: *



(a) Alphabet B3



(b) $W_\Gamma(n)$ vs n



(c) $W_\Gamma(n+1)$ vs $W_\Gamma(n)$

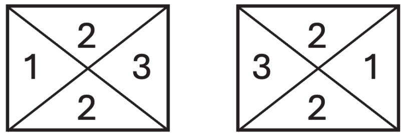
FIG. 12: Alphabet B3 and its associated complexity plots.

highly stable and desirable growth characteristics, fully supporting the minimal oscillation condition $|\omega(z)| \ll 1$ in Equation (11).

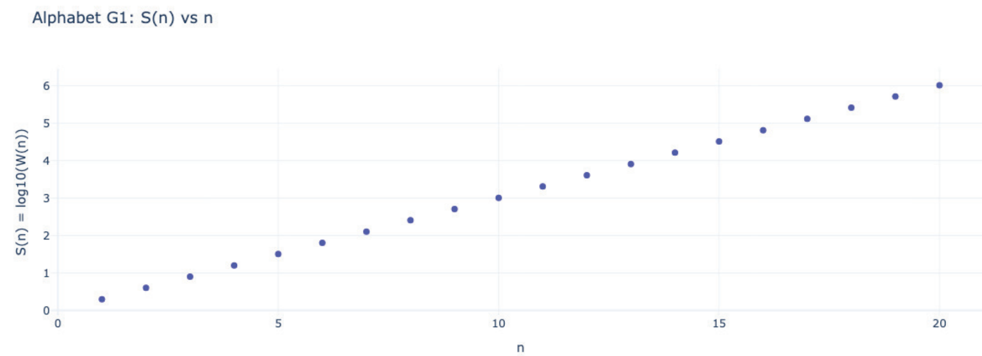
Figure 18 presents the complexity analysis for Alphabet G3, another example of a "good" alphabet. Figure 18(b) illustrates the growth of $S_\Gamma(n)$ with n, showing a consistent monotonic increase in entropy, indicative of stable combinatorial expansion. The core analysis of growth monotonicity, depicted in Figure 18(c) plotting $S_\Gamma(n+1)$ against $S_\Gamma(n)$, reveals a positive linear correlation, with data points closely adhering to the regression line. As per Table I, Alphabet G3 exhibits regression coefficients $c_0 = 1.54$ and $\gamma = 0.810$. The γ value, close to unity, consistently signifies a robust, near-linear growth in the logarithmic domain, characteristic of predictable complexity. Critically, the Kendall's Tau coefficient $\tau = 1.00$ confirms a perfect positive monotonic relationship, where $S_\Gamma(n+1)$ consistently increases with $S_\Gamma(n)$. This perfect monotonicity, combined with the observed linearity, firmly categorizes Alphabet G3 as having highly stable and desirable growth characteristics, fully supporting the minimal oscillation condition $|\omega(z)| \ll 1$ in Equation (11).

Figures 20 and 22 depict the complexity analysis for Alphabets B1 and B2, respectively. Despite originating from distinct alphabet structures, their $S_\Gamma(n)$ plots and regression behaviors are empirically identical, classifying them as "bad" alphabets. Figure 20(b) (and similarly for B2) illustrates that $S_\Gamma(n)$ still shows an increasing trend with n, but with noticeable plateaus and less uniform steps, hinting at a less stable combinatorial growth. The key divergence from "good" alphabets is evident in the $S_\Gamma(n+1)$ vs $S_\Gamma(n)$ plot (Figure 20(c)). While a general linear trend can be

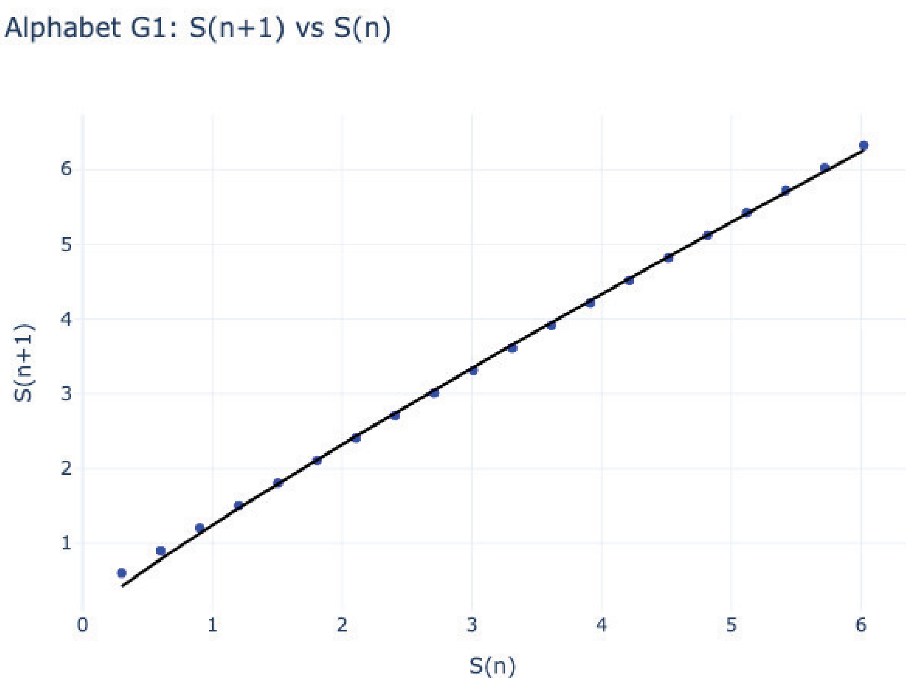
FIG. 13: *



(a) Alphabet G1



(b) $S_\Gamma(n)$ vs n

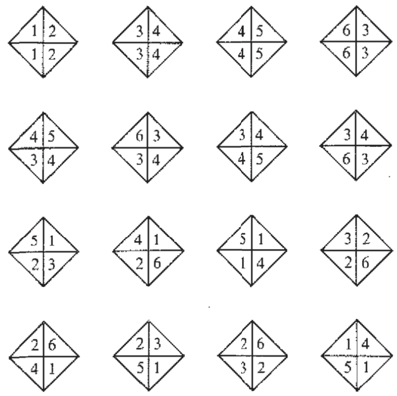


(c) $S_\Gamma(n+1)$ vs $S_\Gamma(n)$

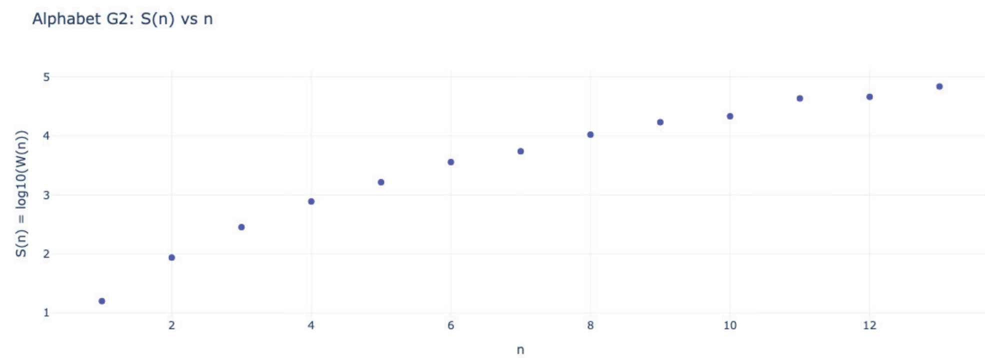
FIG. 14: Alphabet G1 and its associated entropy plots.

observed, there is significant dispersion of data points around the regression line, especially at higher $S_\Gamma(n)$ values, indicating structural inconsistencies and a less predictable relationship between successive entropy values. From Table I, both Alphabets B1 and B2 share coefficients: $c_0 = 1.75$ and $\gamma = 0.550$. The γ value, being significantly lower than unity, indicates a weaker, sub-linear growth in the logarithmic domain, suggesting that the relative increase in complexity diminishes as n grows. This contrasts sharply with the stable growth seen in "good" alphabets. Crucially, the Kendall's Tau coefficient $\tau = 0.72$ is markedly less than 1.00. This value confirms a positive correlation but

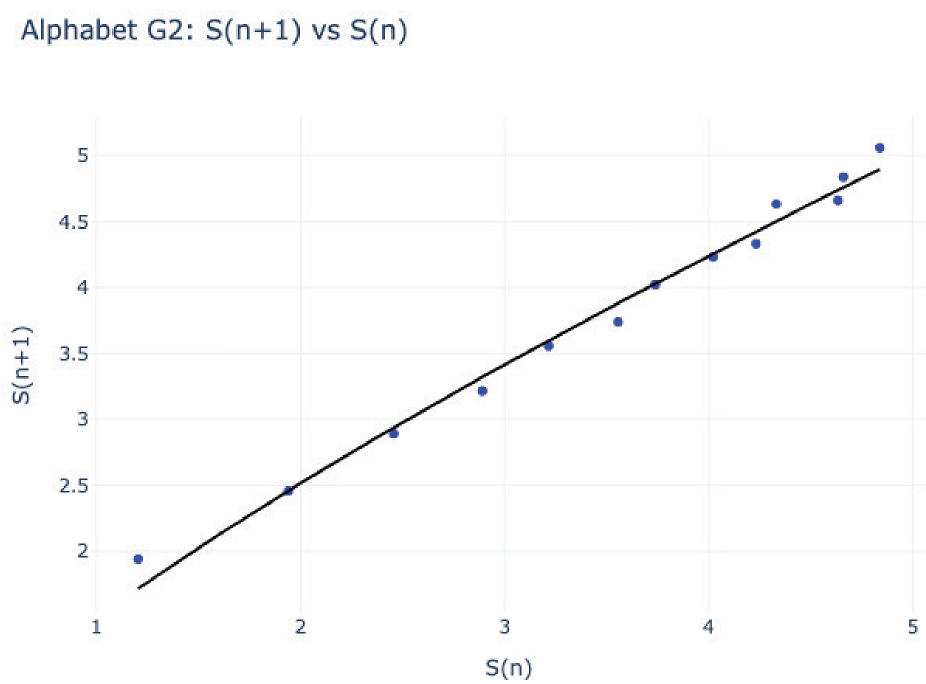
FIG. 15: *



(a) Alphabet G2



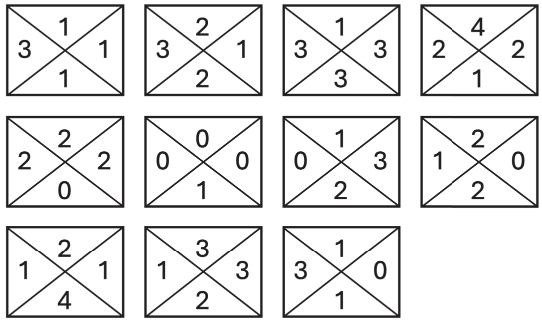
(b) $S_\Gamma(n)$ vs n



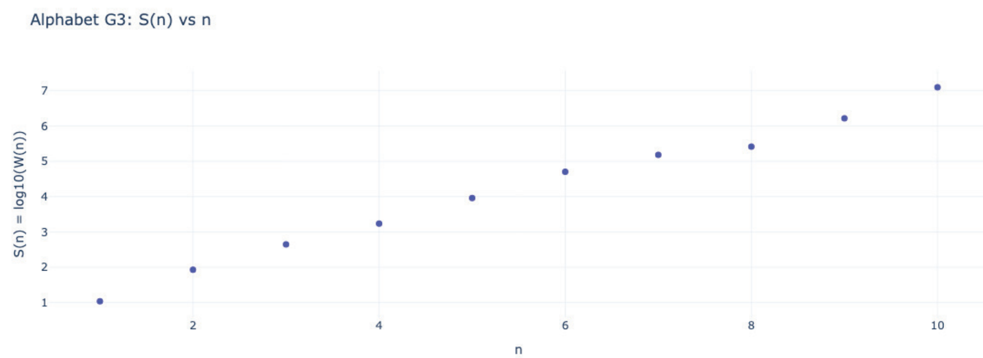
(c) $S_\Gamma(n+1)$ vs $S_\Gamma(n)$

FIG. 16: Alphabet G2 and its associated entropy plots. While the behavior is not linear, it exhibits a monotonically increasing trend.

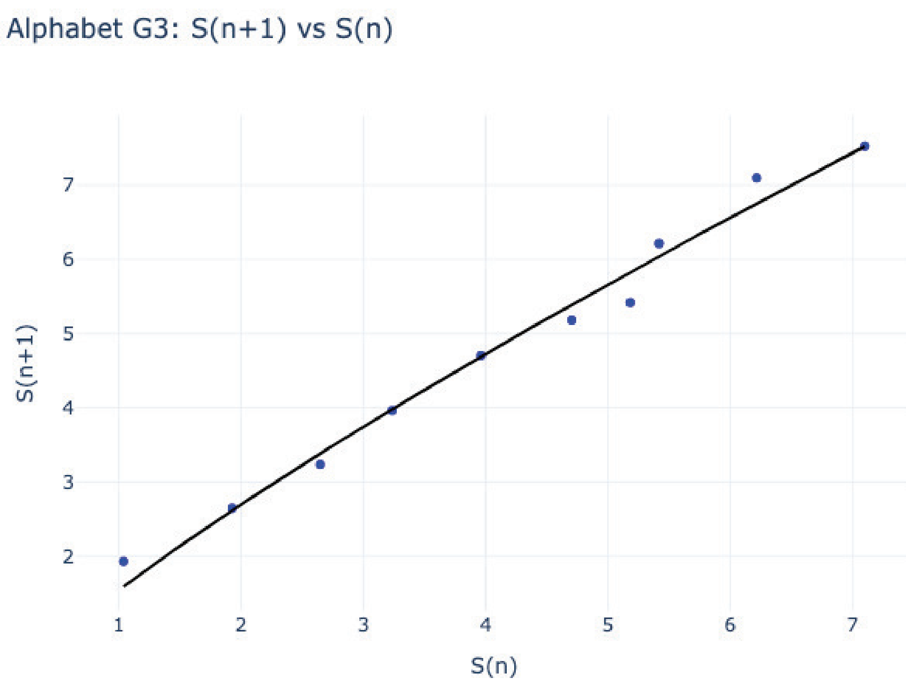
FIG. 17: *



(a) Alphabet G3



(b) $S_\Gamma(n)$ vs n



(c) $S_\Gamma(n+1)$ vs $S_\Gamma(n)$

FIG. 18: Alphabet G3 and its associated entropy plots.

explicitly indicates the presence of monotonic inversions or plateaus, where $S_\Gamma(n+1)$ does not consistently increase with $S_\Gamma(n)$. This departure from perfect monotonicity is the defining characteristic of "bad" alphabets, reflecting the limitations in their combinatorial expansion and highlighting the non-trivial oscillations of $\omega(z)$ in Equation (11).

Figure 24 presents the complexity analysis for Alphabet B3, which also falls into the category of "bad" alphabets. Figure 24(b) illustrates the behavior of $S_\Gamma(n)$ with n. Similar to B1 and B2, while an overall increasing trend is observed, the growth is not perfectly smooth, showing some irregularities in the rate of entropy increase. More critically, the $S_\Gamma(n{+}1)$ vs $S_\Gamma(n)$ plot in Figure 24(c) demonstrates significant deviation from a strict linear relationship, exhibiting a noticeable scatter of data points around the regression line. This scatter is particularly pronounced at higher $S_\Gamma(n)$ values, indicating a less predictable and less stable progression of complexity. From Table I, Alphabet B3 has regression coefficients $c_0 = 1.86$ and $\gamma = 0.596$. The γ value, significantly less than unity, confirms a sub-linear growth in the logarithmic domain, implying that the incremental increase in complexity diminishes with increasing sequence length, a hallmark of "bad" alphabets. The most telling characteristic is the Kendall's Tau coefficient $\tau = 0.79$, which, while higher than B1/B2, remains significantly below 1.00. This value definitively indicates the presence of monotonic violations, where $S_\Gamma(n+1)$ does not consistently increase with $S_\Gamma(n)$, reflecting the inherent structural inconsistencies and the non-negligible oscillations of $\omega(z)$ in Equation (11).

The comprehensive analysis of alphabets, based on their entropy growth $S_\Gamma(n)$, reveals a clear distinction in their combinatorial behavior. The results for "bad" alphabets, exemplified by B1, B2, and B3, consistently indicate a sub-linear growth in entropy (characterized by $\gamma < 1$) and a loss of perfect monotonicity (indicated by $\tau < 1.00$). This behavior is consistent with an inherent limitation in their ability to efficiently tile the entire space, often stemming from the absence of superposable structures. This diminished growth rate and structural inconsistency is precisely corroborated by the specific c_0, γ, and τ values presented in Table I, and visibly reinforced by the corresponding $S_\Gamma(n+1)$ vs $S_\Gamma(n)$ plots. Conversely, "good" alphabets (G1, G2, G3) exhibit near-linear growth in the logarithmic domain ($\gamma \approx 1$) and perfect monotonicity ($\tau = 1.00$), demonstrating stable and predictable complexity expansion. Consequently, these empirical observations provide a robust basis for distinguishing between alphabets with *good* and *bad* tiling properties based on their entropy scaling and monotonic characteristics.

For bad alphabets, the numerical fits of the plots can still be described by Eq. (11). On the other hand, the condition in Eq. (12) is violated for bad alphabets, namely the oscillations around the power law behavior are large.

Thus, the *partial decidability protocols* proposed in the present manuscript is based on this pronounced difference. In simple terms, the protocol reads as follows:

Step 1) Compute numerically as many values as possible of the function $W_\Gamma(n)$ (subject to the available resources, the ideal situation being having $n_{\max}$ larger than q of, at least, a factor of 2 or 3: see the comments below Eq. (6)) for the alphabet Γ of interest.

Step 2) Due to the large amount of combinations, it is better to use the entropy $S_\Gamma(n)$ as main variable .

Step 3) Produce the plot $P = (X, Y) = (S_\Gamma(n), S_\Gamma(n+1))$ as described in the previous section.

Step 4) Construct the best fit of $S_\Gamma(n+1) = f_\Gamma(S_\Gamma(n))$ and compute the employ Kendall's Tau (τ).

IV. PHYSICAL INTERPRETATION IN TERMS OF DISCRETE CHAOS

As it has been already emphasized in the introduction, there is a deep connections between deterministic chaos and undecidability since the Chaitin's results in [18]. The "coexistence" of determinism and undecidability (see, for instance, the detailed analysis in [19] [20] [21]) implies that undecidability can manifest itself in dynamical systems manifesting chaotic behavior. The idea to use the second protocol instead of the first (simpler) protocol is that the second one reveals interesting informations on the transition from good to bad alphabets. The qualitative explanation of the effectiveness of the present approach is based on the transition to chaos in discrete mappings of logistic type (see, for a detailed review on discrete chaos, [17]).

Let us interpret

$$S_\Gamma(n+1) = f_\Gamma(S_\Gamma(n)) \ , \tag{13}$$

as a discrete dynamical system (a possible choice discussed in the previous section is $f_\Gamma(z) = c_0 z^\gamma (1 + \omega(z))$ although the results in the present sections apply to generic form of $f_\Gamma(z)$).

In particular, if $f_\Gamma(z)$ *is monotone increasing*[4] *(namely,* $\frac{d}{dz} f_\Gamma(z) > 0$*)* the solutions of the above dynamical system

[4] According to the present numerical results, this corresponds (when the parametrization $f_\Gamma(z) = c_0 z^\gamma (1 + \omega(z))$ is used) to the case in which $|\omega(z)| \ll 1$.

will tend to the solutions of the simpler (non-chaotic) dynamical system here below:

$$S_\Gamma(n+1) = f_\Gamma^{(0)}(S_\Gamma(n)) \ , \quad f_\Gamma^{(0)}(z) = c_0 z^\gamma \ , \tag{14}$$

according to which $W_\Gamma(n)$ grows exponentially with n with subexponential corrections.

On the other hand, if $f_\Gamma(z)$ *is not monotone increasing*[5] *(namely,* $\frac{d}{dz} f_\Gamma(z)$ vanishes and changes sign, generically, more than once*) and local maxima and minima appear*, the dynamical system in Eq. (13) can enter into a chaotic phase. In particular, the more peaked is the local maximum, the closer is the shape of $f_\Gamma(z)$ to a logistic map in the chaotic phase. Thus, although the explicit analytic form of $f_\Gamma(z)$ is not available, the present results show that the maps $f_\Gamma(z)$ associated to good alphabets do not manifest chaotic tendency while the ones associated to bad alphabets manifest a clear chaotic tendency.

In particular, in the terminal region of the experimental data for non-tiling ("bad") alphabets, one can observe a departure from sustained growth, manifesting as a decreasing trend in the relationship between successive values. While initial iterations may exhibit increasing values of $S(n+1)$ with respect to $S(n)$, the final experimental points reveal a clear inversion, indicating that larger values of $S(n)$ are associated with smaller subsequent values of $S(n+1)$. This decay suggests an inherent constraint within these alphabets that limits their expansive behavior.

This eventual decrease in $S(n+1)$ as $S(n)$ increases can be qualitatively described by a functional form reminiscent of a downward-opening parabola. Although not necessarily an exact fit, a quadratic model of the type

$$S(n+1) \approx S_0 - a[S(n) - b]^2 \ , \tag{15}$$

where S_0, $a > 0$, and b are constants, captures the essence of this behavior. This form illustrates how, beyond a certain point related to b, increasing $S(n)$ leads to a reduction in $S(n+1)$, ultimately resulting in the observed decay. This tendency towards a decreasing relationship at larger values of $S(n)$ appears to be a distinguishing feature of alphabets that are unable to tile the plane: the similarity with the logistic map is manifest. We will come back on this very interesting issue in a future publication.

V. QUANTIFYING "GOODNESS" AND "BADNESS" OF ALPHABETS

In this section, we will discuss how the physical arguments in the previous sections are actually confirmed by a statistical analysis of the numerical data. In complex systems involving sequential or time-dependent data, quantifying the relative quality of different datasets presents significant analytical challenges. This section examines how correlation measures can serve as effective tools for evaluating monotonic relationships within our alphabets, allowing us to distinguish between what we term "good" and "bad" behaviors.

When analyzing correlation in datasets with extreme values and seeking to identify monotonic trends rather than strictly linear relationships, the choice of correlation coefficient becomes crucial. While the Pearson correlation coefficient is commonly employed in statistical analysis, its limitations make it suboptimal for our specific requirements.

The Pearson correlation coefficient (r) measures the linear relationship between variables:

$$r = \frac{\mathrm{cov}(X, Y)}{\sigma_X \sigma_Y}$$

However, this measure presents several drawbacks for our analysis. Pearson's coefficient assumes linearity in relationships, is highly sensitive to extreme values and outliers, requires normally distributed data, and can be significantly influenced by the scale and magnitude of observations. These characteristics make it inadequate for our datasets, which contain extreme values and where we are primarily concerned with monotonic trends rather than linear relationships.

Kendall's Tau (τ), a non-parametric measure of rank correlation, offers a more suitable alternative for our analysis:

$$\tau = \frac{(\text{Number of concordant pairs}) - (\text{Number of discordant pairs})}{\frac{1}{2}n(n-1)}$$

---

[5] According to the present numerical results, this corresponds (when the parametrization $f_\Gamma(z) = c_0 z^\gamma (1 + \omega(z))$ is used) to the case in which $|\omega(z)|$ *is not small compared to 1.*

Kendall's Tau evaluates correlation based on the concordance of pairs—whether the ranks of paired observations move in the same direction—rather than the magnitude of differences. This approach provides several advantages for our specific analytical needs:

First, Kendall's Tau is particularly robust when dealing with datasets containing outliers or extreme values, as is the case with several of our alphabets. Because it focuses on the relative ordering of data points rather than their actual values, the coefficient is less affected by unusually large or small observations.

Second, Kendall's Tau excels at detecting monotonic relationships without assuming linearity. Our primary interest lies in determining whether the regression patterns across different alphabets consistently increase or decrease, making monotonic behavior more relevant than strict linearity.

Third, the coefficient maintains its reliability even with smaller sample sizes, which is beneficial when analyzing rarer alphabetic patterns.

Kendall's Tau has demonstrated its utility across numerous scientific disciplines. In environmental science, researchers have employed it to detect monotonic trends in climate data where extreme weather events might skew other correlation measures [22]. In economics, it has proven valuable for analyzing market behaviors when dealing with volatile price movements [23]. Researchers regularly use Kendall's Tau when examining relationships between particulate matter progression, particularly when distributions are non-normal [24].

Our methodology involves applying Kendall's Tau to measure the monotonic trends present in different alphabets. For "good" alphabets, we observe high τ values, indicating strong monotonic relationships consistent with theoretical expectations. Conversely, "bad" alphabets exhibit lower τ values, reflecting weaker monotonic behavior and greater dispersion in the data.

By employing Kendall's Tau as our primary correlation measure, we establish a robust framework for distinguishing between alphabets that exhibit consistent, predictable behavior (high τ values) and those that demonstrate erratic patterns (low τ values). This approach provides an objective metric for quantifying the relative "goodness" or "badness" of different alphabets, facilitating more rigorous comparative analysis and more reliable conclusions about their underlying properties.

VI. RESUMING THE RESULTS OF THE PRESENT WORK

Here we will resume what we have done in the present work. We have associated to any alphabet Γ a temperature, an entropy and a partition function. The requirement of a good thermodynamical behavior provides one with a very nice criteria (sufficient conditions) ensuring that Γ can tile the plane. The simplest of all is the first protocol which only uses the fact that in most of physically reasonable systems the degeneracy of the energy level is an increasing function of the energy. The second protocol which has been defined discloses a nice analogy between the transition from good to bad alphabets and the transition from regular to chaotic behavior in discrete mapping of logistic type. In order to use these idea in practice, the Kendall's Tau must be used. It seems that the present strategy has a quite wide range of possible applications.

A. What we have not done

As far as the first protocol is concerned, the idea is physically simple and correct but in practice one cannot compute $W_{\Gamma}(n)$ for any n so that one has to stop at some finite $n_{\max}$. We do not have a rigorous answer to the question of how large $n_{\max}$ should be (compared to q) in order to be sure that the alphabet under examination is a good one. For instance, we cannot exclude the following situation. It could happen that we have the numerical verification of Eq. (4) for (1, 2, 3,.., $q \cdot 10^{10^q}$) but then, when $n = 1 + q \cdot 10^{10^q}$, the condition in Eq. (4) is violated. On the other hand, in the examples we have examined (where we have periodic tilings, aperiodic self-similar tiling, tilings which are neither periodic nor self-similar) it seems that if $n_{\max}$ is 2 or 3 times q is large enough. Similar considerations hold for the second protocol.

As far as the similarity of the transition from good to bad alphabets and of the transition from regular to chaotic behavior in discrete mapping of logistic types, we have not shown rigorously that bad alphabets are always associated to chaotic discrete mappings while good alphabet to regular ones. The difficulty lies in the fact that the analytic form of $f_{\Gamma}(z)$ cannot be computed and so one has to use numerical fits of the mapping $f_{\Gamma}(z)$. The presence or absence of chaotic behavior may depend on the family of functions which are chosen to do the fit. However, the connections between deterministic chaos and undecidability (discussed in details in [18] [19] [20] [21] and references therein) lead to the fact that the existence of deterministic chaos in dynamical systems can be related to a sort of Godelian undecidability rather than to the typical numerical untractability. Since the Wang tiling problem is also undecidable, it is reasonable to assume that discrete mappings associated to Wang alphabets manifest their undecidability through chaotic behavior (consistently with [19] [20] [21]).

We will come back on these interesting issues in a future publication.

VII. CONCLUSIONS AND PERSPECTIVES

In the present manuscript we have proposed two partial decidability protocols for the Wang tilings problem (which is the prototype of the undecidable problem in combinatorics and statistical mechanics). The idea is to define effective entropy and temperature associated to any alphabet Γ (together with the corresponding partition function). A subclass of *good alphabets* can be identified by requiring, basically, a good thermodynamical behavior. This proposal has been tested successfully with the known available good alphabets (which produce periodic tilings, aperiodic but self-similar tilings as well as tilings which are neither periodic nor self-similar). From the theoretical physics viewpoint, it is a very intriguing result to be able to reduce the undecidable instances of the Wang tiling problem using sound arguments from statistical mechanics. The present analysis also shows that the transition from good to bad alphabets is very similar to the transition from regular to chaotic behavior in discrete mappings of logistic type.

From the practical viewpoint, the present results are quite powerful. The computational challenges encountered in scaling Wang tilings, particularly the rapid increase in effort required to expand the tiling plane with larger alphabets, suggest that determining whether a given alphabet can tile the infinite plane may be an inherently difficult problem, potentially residing within the NP-hard class. The observed distinction between "good" alphabets, which allow for tiling, and "bad" alphabets, which exhibit eventual decreasing trends in their associated mappings, further hints at a fundamental difference in their complexity. While this study does not definitively resolve the P vs NP problem, the apparent difficulty in efficiently finding a tiling solution, even when a valid configuration can be readily verified, aligns with the prevailing conjecture that P $\neq$ NP, implying that the Wang tiling problem could represent a scenario where finding a solution is significantly more computationally demanding than checking its correctness.

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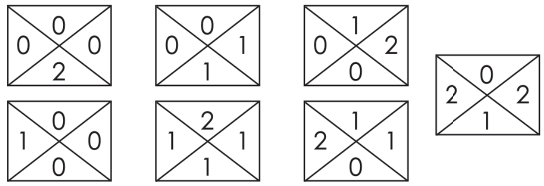
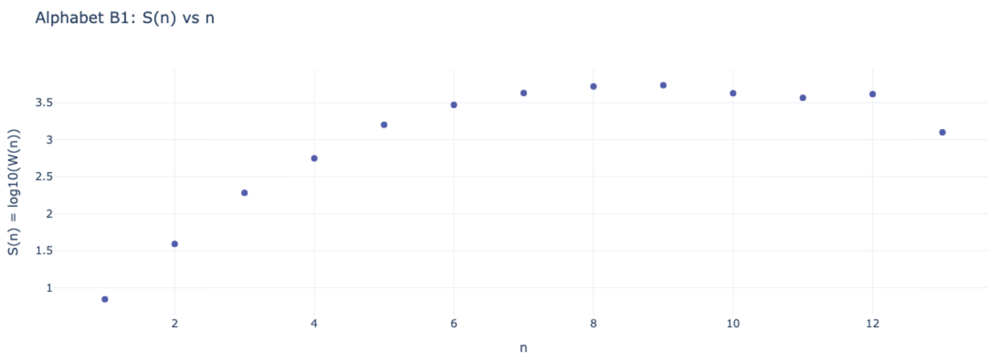
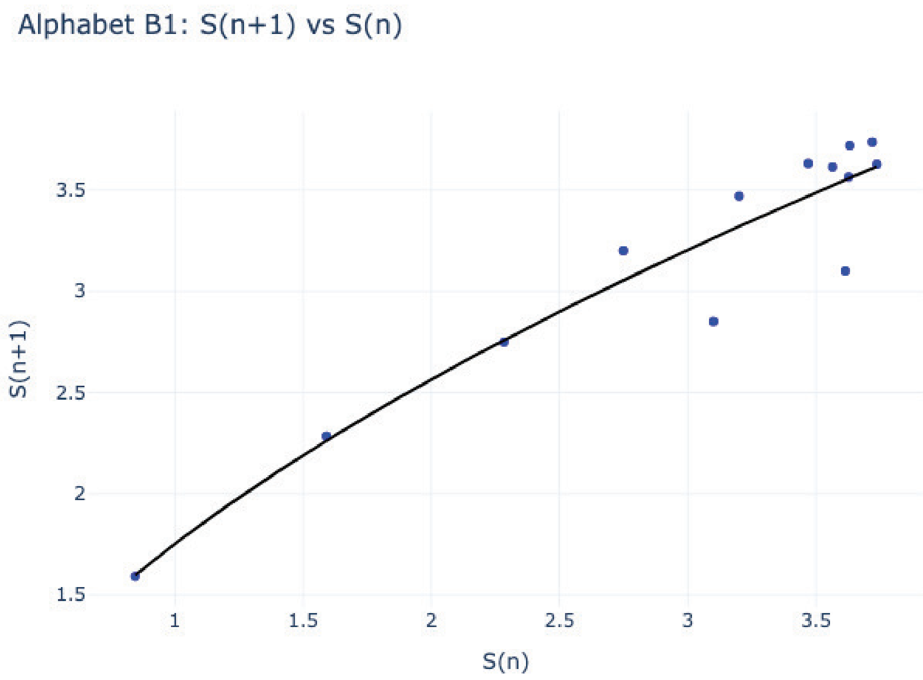
FIG. 19: *



(a) Alphabet B1



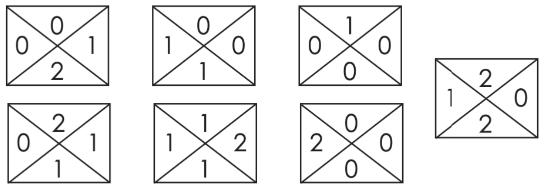
(b) $S_\Gamma(n)$ vs n



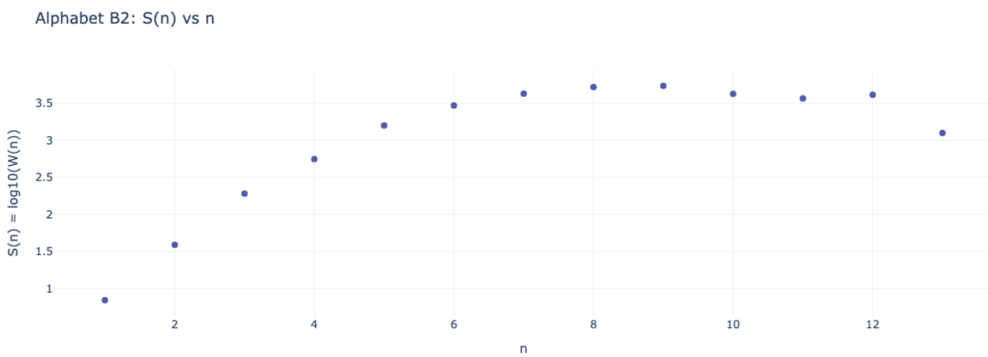
(c) $S_\Gamma(n+1)$ vs $S_\Gamma(n)$

FIG. 20: Alphabet B1 and its associated entropy plots.

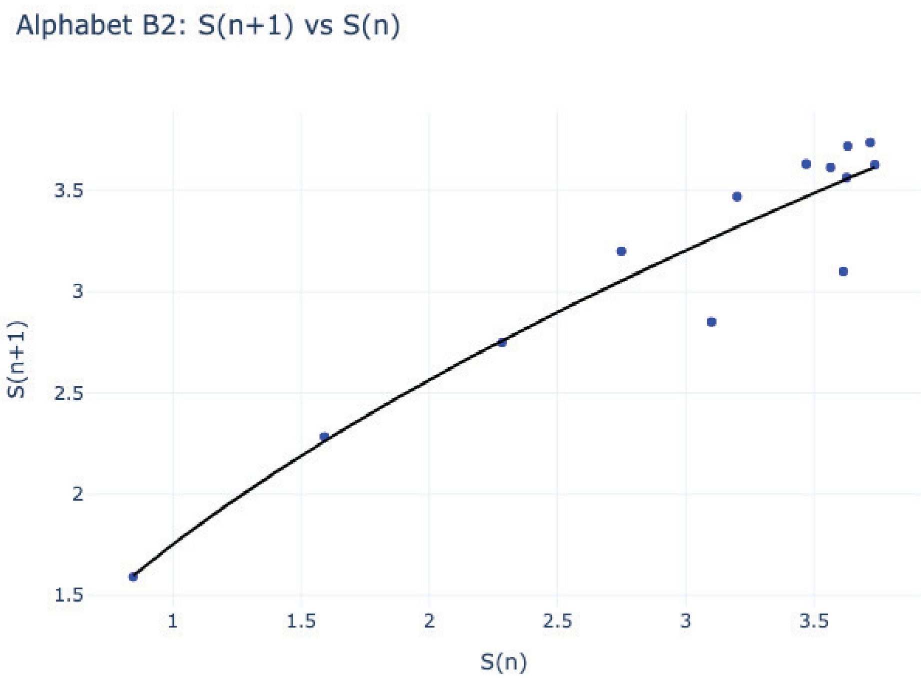
FIG. 21: *



(a) Alphabet B2



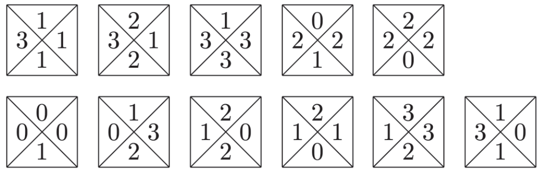
(b) $S_\Gamma(n)$ vs n



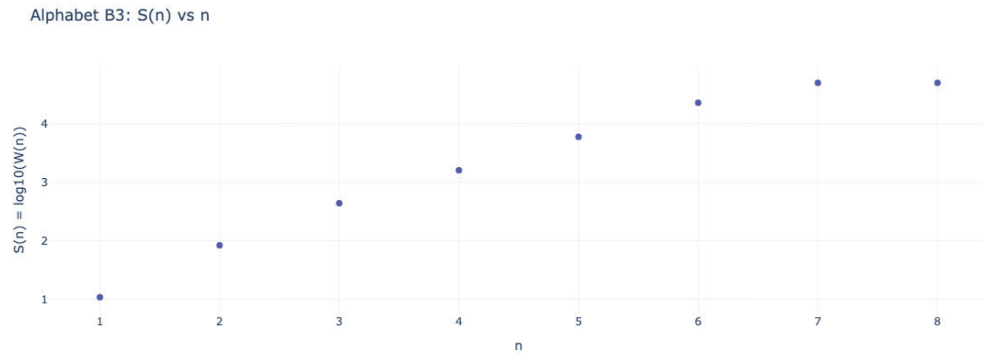
(c) $S_\Gamma(n+1)$ vs $S_\Gamma(n)$

FIG. 22: Alphabet B2 and its associated entropy plots.

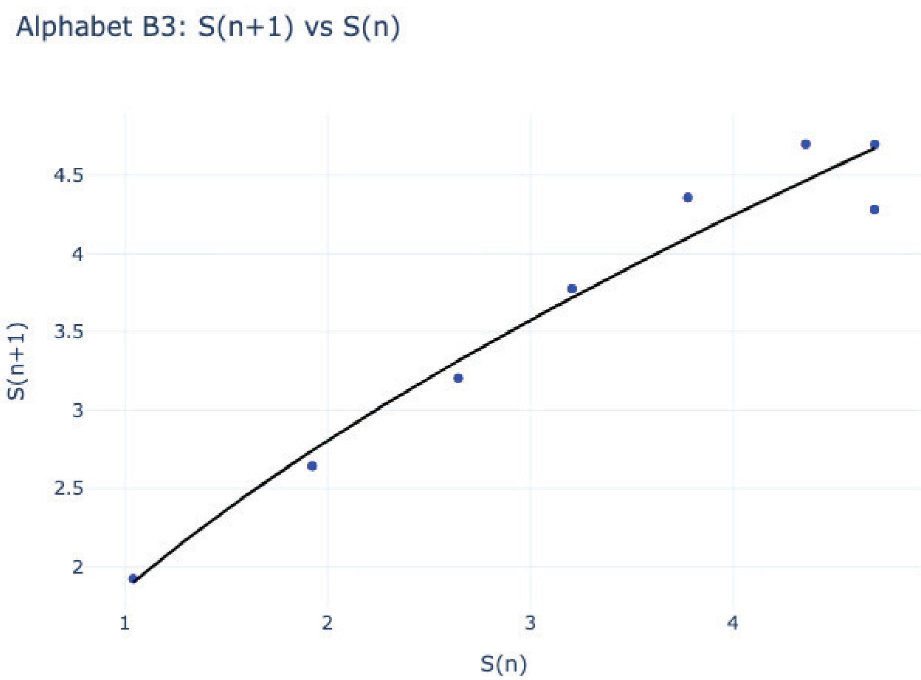
FIG. 23: *



(a) Alphabet B3



(b) $S_\Gamma(n)$ vs n



(c) $S_\Gamma(n+1)$ vs $S_\Gamma(n)$

FIG. 24: Alphabet B3 and its associated entropy plots.