Confronting IR divergences in de Sitter QFT

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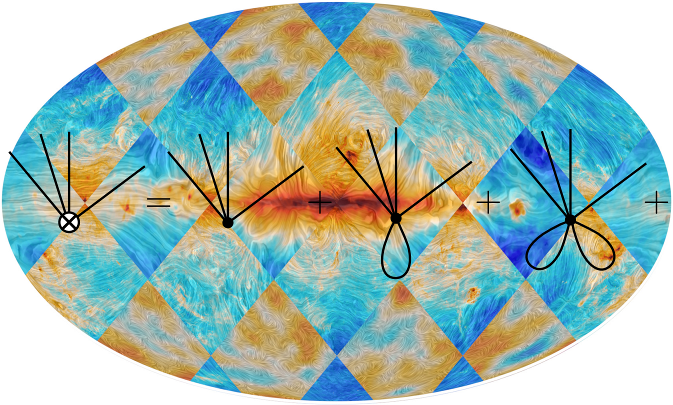


Put a scalar QFT on dS and just wait...

Related issues:

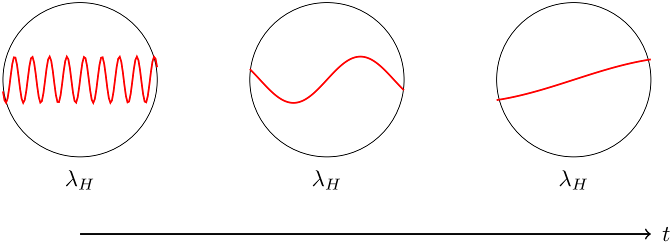
- ⋆ dS stability
- ⋆ cmb/lss phenomenology
- ⋆ PT during inflation
- ⋆ DE?

The problem

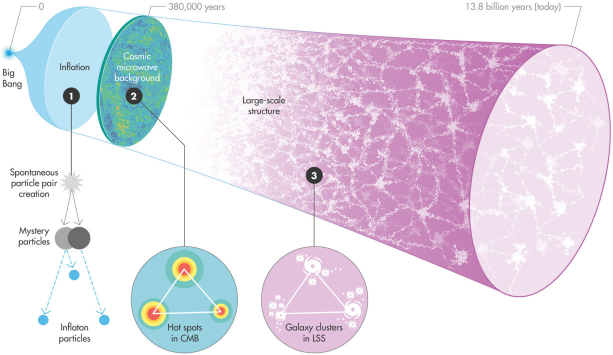
Probe scalar in the Poincare patch of (rigid) de Sitter space ($H =$ const):

$$\varphi,\ V(\varphi) \quad \Big| \quad ds^2 = a^2(\tau)(-dt^2 + dx^2),\ \ a = (H\tau)^{-1}.$$

Wavelengths stretch $\lambda = \ell a$, wavevectors shrink $p = k/a$.



- ⋆ Q: What is the statistics (correlators/PDF) of long modes? (random process in an open system)
- ⋆ Methods: QFT (Feynman graphs) = Stochastic (Langevin eq)



$$\boxed{\mathrm{P}(\varphi) \to \mathrm{P}(\delta T) \to \mathrm{P}(\delta_{\mathrm{g}}) \;,\; \mathrm{P}(\delta_{\mathrm{pbh}})}$$

$\mathcal{V}(\varphi)$ on dS

$$S = \int \mathrm{d}^3 x \, \mathrm{d}\tau \, a^2 \left[\frac{1}{2} \dot{\varphi}^2 - \frac{(\nabla \varphi)^2}{2} - a^2 \mathcal{V}(\varphi) \right]$$

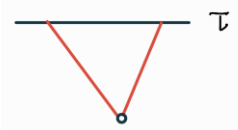
$$\mathcal{V}(\varphi) = \sum_{n=2}^{\infty} \frac{\lambda_n^{\mathrm{bare}}}{n!} \varphi^n$$

We will be interested in single-point statistics (i.e. histograms marginalised over $\boldsymbol{x}$)

$$\langle \varphi^n \rangle = \langle \varphi(\boldsymbol{x}_1) \cdots \varphi(\boldsymbol{x}_n) \rangle \Big|_{\boldsymbol{x}_1 = \cdots = \boldsymbol{x}_n} \propto \lambda_n^{\mathrm{obs}}$$

UV/IR structure of correlators

The simplest correlator is the (free theory) one-point variance:



$$\sigma^2 = H^2 \int_0^\infty \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \frac{1}{k^3} \left(1 + (k\tau)^2\right)$$

or, in physical momentum $p \equiv Hk\tau$

$$\sigma^2 = \left(\frac{H}{2\pi}\right)^2 \int_0^\infty \frac{\mathrm{d}p}{p} \left(1 + \frac{p^2}{H^2}\right)$$

UV divergence: $\propto p^2$

IR divergence: $\propto \ln p$

Cure for external legs: restrict to long modes

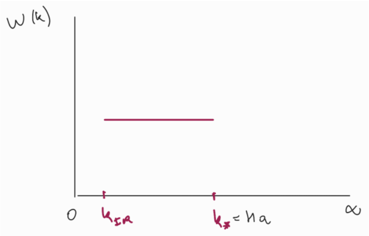
Long-mode statistics

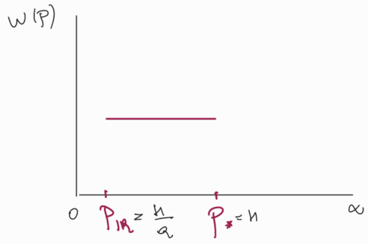
Long-short split: $\varphi = \varphi_S + \varphi_L$:

$$\varphi_L(x,t) \equiv \hat{\boldsymbol{L}}\,\{\varphi(x,t)\} = \int_k e^{ikx} W(k) \tilde{\varphi}_k(t)$$

$$W(k) = \theta(k_*(t) - k) \times \theta(k - k_*(t_\mathrm{i})) \; || \; W(p) = \theta(H - p) \times \theta(p - k_*(t_\mathrm{i})/a)$$

Cut off external legs at $k_*(t_\mathrm{i}) \equiv k_\mathrm{IR}$. This is our choice!





Secular structure of correlators

Now

$$\sigma_L^2(t) = \left(\frac{H}{2\pi}\right)^2 \int_{H/a}^{H} \frac{\mathrm{d}p}{p} = \frac{H^3 t}{4\pi^2}$$

We regularized but we now have a secular behaviour.

Linear modes independent $\rightarrow$ variance additive $\rightarrow$ blows up asymptotically (secular).

Cure: resummation (e.g. stochastic formalism)

What does the Fokker-Planck resum?

Edgeworth expansion of the PDF truncated to linear order in the interactions (single-vertex diagrams):

$$\rho(\varphi, t) = \frac{e^{-\frac{1}{2}\frac{\varphi^2}{\sigma^2(t)}}}{\sqrt{2\pi\sigma^2(t)}}\left[1 + \Delta(\varphi, t)\right]$$

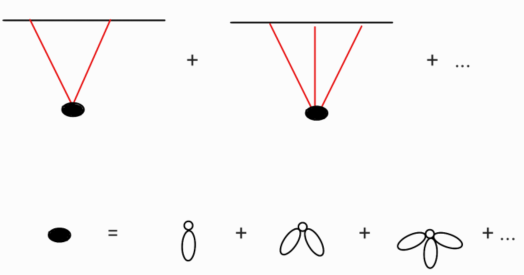
with

$$\Delta(\varphi, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left\langle \varphi^n(t) \right\rangle_c}{\sigma^n} \mathrm{He}_n(\varphi/\sigma)$$

$\boxed{\text{Perturbative QFT} = \text{Gaussian variables} =}$

$\boxed{\text{Feynman rules} = \text{Wick's theorem}}$

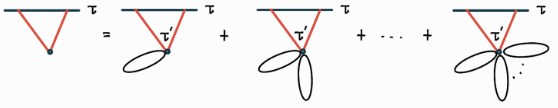
FP resums single-vertex all-loop correlators into an Edgeworth expansion of the PDF



The loop time-dependence matters!

Daisy loops

Do loops change the secular growth?



$$\mathcal{V}(\varphi) = \lambda_4^{\rm bare}\varphi^4 + \lambda_6^{\rm bare}\varphi^6 + \cdots\,, \quad (\text{e.g. axion } \mathcal{V} \propto 1 - \cos\varphi/f)$$

Recall that loops are virtual processes; they control the validity.

Cure for internal legs: UV & IR cutoffs or dim reg + renormalization

Now there is a choice to be made! Comoving vs physical IR cutoff.

Is $t_{\rm i}$ part of the theory?

Define the regularized free-theory variance as

$$\sigma_0^2 = \left(\frac{H}{2\pi}\right)^2 \int_{\Lambda_{\rm IR}}^{\Lambda_{\rm UV}} \frac{\mathrm{d}p}{p} = \left(\frac{H}{2\pi}\right)^2 \ln \frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}$$

The way we cut off loops matters. Recall $p = k/a(t)$. A constant comoving cutoff implies $\Lambda_{\rm IR} = \Lambda_{\rm IR}(t)$ and $\sigma_0 = \sigma_0(t)$ (just like the observable variance $\sigma_L(t)$). A constant physical cutoff implies $\Lambda_{\rm IR} = $ const and $\sigma_0 =$ const. This affects the dynamics of the PDF (which is exactly what the stochastic formalism attempts to describe). Finally, time dependence breaks dS boosts.

A triad of equivalent statements:

1) Loop infrared cutoff Starobinsky '86; Tsamis & Woodard '05; +++
2) Stochastic formalism Starobinsky & Yokoyama '94; Tsamis & Woodard '05; +++
3) Breaking of dS isometries by massless zero mode Allen & Folacci '87

A comoving loop cutoff leads to an FP equation that admits an equilibrium solution:

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2 \rho}{\partial \varphi^2} + \frac{1}{3H}\frac{\partial}{\partial \varphi}\left(\rho \mathcal{V}'_{\rm ren}\right) \quad || \quad \left[\rho_\infty \propto e^{-\mathcal{V}/H^4}\right]$$

Starobinsky & Yokoyama '94; Tsamis & Woodard '05; +++

A physical loop cutoff leads to an FP equation that admits an equilibrium solution:

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial \varphi^2}\left[\rho\left(1 - \frac{2Ht}{3H^2}\mathcal{V}''_{\rm ren}\right)\right] + \frac{1}{3H}\frac{\partial}{\partial \varphi}\left(\rho \mathcal{V}'_{\rm ren}\right)$$

Physical cutoff: loops renormalise the vertex

Comoving cutoff: Loops enhance secular growth

To sum-up

- ⋆ Statistics of long scalar modes on dS
- ⋆ resummation of Feynman diagrams $=$ stochastic formalism
- ⋆ Important details :

  Physical vs comoving loop regulator: cumulative diffusion
  Need for resummation in $\mathcal{V}$ to reach equilibrium

Thanks!

Let us first choose a physical cutoff (dS invariant regulator). Then

$$\sigma_0^2 = \left(\frac{H}{2\pi}\right)^2 \int_{\Lambda_{\rm IR}}^{\Lambda_{\rm UV}} \frac{\mathrm{d}p}{p} = \left(\frac{H}{2\pi}\right)^2 \ln \frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}$$

SK, *in-in*, *WFU*: n-pt functions loop-resummed to:

$$\left\langle \varphi^n(\tau) \right\rangle_c = -\frac{8\pi}{H^4} \mathrm{Im}\left\{ \int_0^\infty \mathrm{d}x \int_{\tau_{\rm i}}^{\tau} \mathrm{d}\bar{\tau}\, \frac{x^2}{\bar{\tau}^4} \lambda_n^{\rm obs} \Big[g(x, \bar{\tau}, \tau) \Big]^n \right\}$$

with g the propagator and

$$\lambda_n^{\rm obs} = \sum_{L=0} \frac{\lambda_{n+2L}^{\rm bare}}{L!} \left(\frac{\sigma_0^2}{2}\right)^L$$

(This is nothing but the coefficients of a Weierstrass transform)

The PDF of long modes

Edgeworth expansion of the PDF truncated to single-vertex diagrams:

$$\rho(\varphi, t) = \frac{e^{-\frac{1}{2}\frac{\varphi^2}{\sigma^2(t)}}}{\sqrt{2\pi\sigma^2(t)}}\left[1 + \Delta(\varphi, t)\right]$$

with

$$\Delta(\varphi, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left\langle \varphi^n(t) \right\rangle_c}{\sigma^n} \mathrm{He}_n(\varphi/\sigma)$$

Upon using

$$\mathcal{O}_\varphi \mathrm{He}_n(\varphi/\sigma) = -n\mathrm{He}_n(\varphi/\sigma)$$

the PDF can be resummed:

$$\Delta(\varphi,\tau) = \frac{8\pi}{H^4}\mathrm{Im}\int_0^\infty \mathrm{d}x \int_{\tau_\mathrm{i}}^{\tau} \mathrm{d}\bar{\tau}\, \frac{x^2}{\bar{\tau}^4}\left(\frac{g(x,\bar{\tau},\tau)}{\sigma^2}\right)^{-\mathcal{O}_\varphi} \underbrace{e^{-\frac{\sigma_0^2}{2}\frac{\partial^2}{\partial\varphi^2}}\mathcal{V}(\varphi)}_{\mathcal{V}_\mathrm{ren}(\varphi)}$$

such that $\rho = \rho_0\left[1+\Delta\right]$

$\boxed{\text{This is the PDF of long modes}}$

Emergence of stochastic dynamics

$k\tau \to 0$ limit:

$$\Delta(\varphi, t) = \frac{Ht}{3H^2}\left(\mathcal{V}''_{\rm ren}(\varphi) - \frac{\varphi}{\sigma^2}\mathcal{V}'_{\rm ren}(\varphi)\right)$$

and we then take a time derivative of the PDF:

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\left[\rho\left(1 - \frac{2Ht}{3H^2}\mathcal{V}''_{\rm ren}\right)\right] + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\rho\mathcal{V}'_{\rm ren}\right)$$

Fokker-Planck equation

Performing the same computation with a comoving IR cutoff:

$$\left\langle \varphi^n(t) \right\rangle_c = -\frac{4\pi^2 n}{3H^4}\sigma^{2n}(t)\sum_L^{\infty}\frac{\lambda_{n+2L}}{(n+L)L!}\left(\frac{\sigma^2(t)}{2}\right)^L$$

leads to a PDF that satisfies the autonomous Fokker-Planck eq:

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2\rho}{\partial\varphi^2} + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\rho\mathcal{V}'_{\rm ren}\right) \quad || \quad \left[\rho_\infty \propto e^{-\mathcal{V}/H^4}\right]$$

Starobinsky & Yokoyama '94; Tsamis & Woodard '05; +++

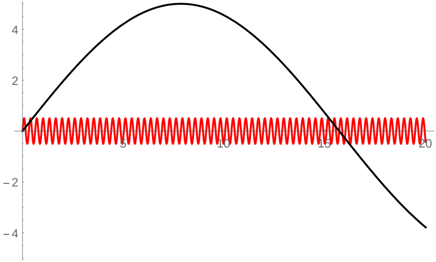
$\boxed{\text{Physical cutoff: loops renormalise the vertex}}$

$\boxed{\text{Comoving cutoff: Loops enhance secular growth}}$

The stochastic approach: Starobinski-Yokoyama 90's

Long modes: nonlin but classical

Short modes: lin but QM



How to model the effect of short modes at large scales?

$$\hat{\varphi}_k = \varphi_{\text{lin}}\hat{a}_k + \varphi^*_{\text{lin}}\hat{a}^\dagger_k \rightarrow \hat{\xi}_k = \varphi_{\text{lin}}(\hat{a}_k - \hat{a}^\dagger_k)$$

The EOM for long modes now becomes a stochastic Langevin equation: for $\Delta t \gg 1/H$,

$$\dot{\varphi}_L = H\hat{\xi}(t) - \frac{1}{3H}\mathcal{V}'_{\rm ren}(\varphi_L)$$

where $\hat{\xi}(t)$ is a Gaussian stochastic noise representing the short-mode bath:

$$\hat{\xi} \equiv H^{-1}\int_k \dot{W}(k)\,\tilde{\varphi}_k\,, \quad \left\langle \hat{\xi}(t_1)\hat{\xi}(t_2)\right\rangle \propto \delta(t_1 - t_2)$$

From this one can straightforwardly get the autonomous FP equation: integrate Langevin

$$\boxed{\varphi_L(t) = \varphi_G(t) - \tfrac{1}{3H}\int \mathrm{d}t'\,\mathcal{V}'_{\rm ren}\left[\varphi_G(t')\right]}$$

compute

$$\langle \varphi_L^n(t)\rangle_c$$

plug in Edgeworth and derive:

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2\rho}{\partial\varphi^2} + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\rho\mathcal{V}'_{\rm ren}\right)$$

Langevin from *in-in*

$\varphi(x,t) = U(t)\varphi_I(x,t)U^\dagger(t)$, where $\varphi_I(x,t)$ is the interaction-picture field and $U = \exp\left\{ -i\int \mathrm{d}t' \int \mathrm{d}^3x'\, \mathcal{V}_{\rm ren}(\varphi_I)\right\}$.

To first order in the potential:

$$\varphi(t) \simeq \varphi_I(t) - \frac{1}{3H}\int \mathrm{d}t'\, \mathcal{V}'_{\rm ren}\left[\varphi_I(t')\right]$$

Now apply $\hat{\boldsymbol{L}}$:

$$\boxed{\varphi_L(t) \simeq \varphi_G(t) - \tfrac{1}{3H}\int \mathrm{d}t'\, \hat{\boldsymbol{L}}\left\{\mathcal{V}'_{\rm ren}\left[\varphi_I(t')\right]\right\}}$$

Starobinski approximation:

$$\hat{\boldsymbol{L}}\left\{\mathcal{V}'_{\rm ren}\left[\varphi_I(t')\right]\right\} = \mathcal{V}'_{\rm ren}(\hat{\boldsymbol{L}}\varphi_I)$$

When does this hold? When we negelct short-long mode correlations. (equivalent to comoving vs physical loop cutoffs)

Computing

$$\langle \varphi_L^n(t) \rangle_c \,,$$

with $\mathcal{V}'_{\rm ren}(\hat{\boldsymbol{L}}\varphi_I)$ leads to

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\rho + \frac{1}{3H}\frac{\partial}{\partial\varphi}\Big(\rho\mathcal{V}'_{\rm ren}\Big)$$

Computing

$$\langle \varphi_L^n(t) \rangle_c \,,$$

with $\hat{\boldsymbol{L}}\left\{\mathcal{V}'_{\rm ren}\left[\varphi_I(t')\right]\right\}$ and following the same steps now leads to

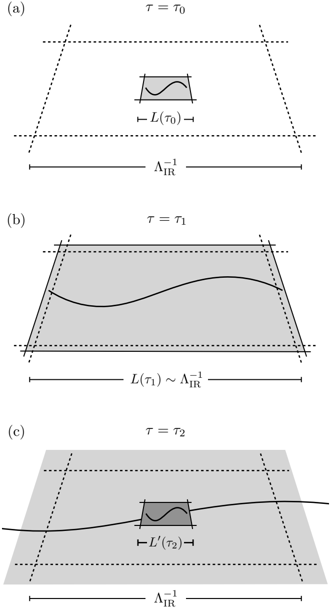
$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\left[\rho\Big(1 - \frac{2Ht}{3H^2}\mathcal{V}''_{\rm ren}\Big)\right] + \frac{1}{3H}\frac{\partial}{\partial\varphi}\Big(\rho\mathcal{V}'_{\rm ren}\Big)$$

Edgeworth expansion

$$\rho(\varphi) = \int \mathrm{d}J\, z(J) e^{-iJ\varphi}$$

Using $z = e^w$, and $w(J) = -\frac{1}{2}\sigma^2 J^2 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \langle \varphi^n \rangle_c J^n$, we obtain

$$\rho(\varphi) = \frac{e^{-\frac{1}{2}\frac{\varphi^2}{\sigma^2}}}{\sqrt{2\pi}\sigma} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{\langle \varphi^{n_1} \rangle_c}{n_1! \sigma^{n_1}} \cdots \frac{\langle \varphi^{n_N} \rangle_c}{n_N! \sigma^{n_N}} \mathrm{He}_{n_1+\cdots+n_N}(\varphi/\sigma)$$



(a)
$\tau = \tau_0$
$L(\tau_0)$
$\Lambda_{\mathrm{IR}}^{-1}$
(b)
$\tau = \tau_1$
$L(\tau_1) \sim \Lambda_{\mathrm{IR}}^{-1}$
(c)
$\tau = \tau_2$
$L'(\tau_2)$
$\Lambda_{\mathrm{IR}}^{-1}$

$$\ddot{\varphi}_k + 3H\dot{\varphi}_k + \frac{k^2}{a^2}\varphi_k + \mathcal{V}' = 0$$

The renormalized potential

Separate the IR modes

$$\varphi = \varphi_S + \varphi_L + \varphi_{\mathrm{IR}}$$

and integrate them out:

$$\mathcal{V}(\bar{\varphi}) = \langle \Psi_{\mathrm{IR}} | \mathcal{V}(\bar{\varphi} + \varphi_{\mathrm{IR}}) | \Psi_{\mathrm{IR}} \rangle$$

with

$$|\Psi_{\mathrm{IR}}\rangle = \int \mathcal{D}\varphi_{\mathrm{IR}} \Psi(\varphi_{\mathrm{IR}}) |\varphi_{\mathrm{IR}}\rangle$$

with $|\varphi_{\mathrm{IR}}\rangle$ an IR field-eigenstate and $|\Psi(\varphi_{\mathrm{IR}})|^2$ the Gaussian. This leads directly to

$$\boxed{\mathcal{V}(\bar{\varphi}) = \mathcal{V}_{\mathrm{ren}}(\bar{\varphi})}$$