Falsifying cosmological models based on a non-linear electrodynamics

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(Dated: November 30, 2017)

Recently, the nonlinear electrodynamics (NED) has been gaining attention to generate primordial magnetic fields in the Universe and also to resolve singularity problems. Moreover, recent works have shown the crucial role of the NED on the inflation. This paper provides a new approach based on a new model of NED as a source of gravitation to remove the cosmic singularity at the big bang and explain the cosmic acceleration during the inflation era on the background of stochastic magnetic field. Although we are mainly interested in early time universe, we also explore whether a NED field can be the origin of the cosmic acceleration. Also, we found a realization of a cyclic Universe, free of initial singularity, due the model for the NED energy density we propose. We find explicit relations for $H(z)$ and perform a MCMC analysis to constrain the NED parameters by using Observational Hubble data (OHD) obtained from differential age (DA) method, which contains 31 data points covering $0 < z < 1.97$; and with the latest SNIa data, that is, the joint-light-analysis (JLA) compilation consisting in 740 data points in the range $0.01 < z < 1.2$. We compute the deceleration parameter $q(z)$ in the range $0 < z < 2$ from the best fit values of the parameters found using OHD and SNIa data and find that $q(z) \to 1/2$ when $z \to \infty$. In addition, both cosmological data predict an accelerating expansion, i.e., the Universe passes of a decelerated phase to an accelerated stage at redshift ~ 0.5 and ~ 0.7 for the OHD and SNIa constraints respectively.

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I. INTRODUCTION

Universe started with a Big Bang, which had a singularity that all the laws of physics would have broken down [1]. Today, the first thing that is known that the two great theories of physics such as quantum mechanics and general relativity mostly work very well, except in some extreme conditions like the Big Bang [2]. There is something which is clearly still missing to explain this singularity [3]. Recently, cosmological models using non-linear electromagnetic fields (NEF) have been gain interest to remove singularity problem of the Universe at the Big Bang and also singularities of curvature invariants [4–8]. The Standard Cosmological Model (SCM) based on Friedmann- Robertson-Walker (FRW) geometry, also known as ΛCDM model, does not produce any solution for the singularity problem at the beginning of the Universe [9]. The SCM has a problem of singularities. If the Maxwell equations are intelligently modified, these singularities can be resolved. There are some cosmological models known as magnetic Universe that has no singularity because of the nonlinear modification of the Maxwell electrodynamics at strong fields such as early universe [10, 12], because of the background of the conformally flat Robertson-Walker metric, but the cost is the break up of the conformal invariance of Maxwell theory [12–21].

Moreover, today it is widely accepted fact among the physicist that the Universe is accelerating. The idea of an accelerating Universe is supported and confirmed by type Ia supernovae and the cosmic microwave background (CMB) [1, 9, 22, 23]. However the reason of the acceleration of the Universe is not entirely clear, nevertheless a number of solutions have been proposed [7, 9, 24–48]. One such solution is to introduce the cosmological constant in the Einstein's field equations. In this scenario, the acceleration of Universe is driven by dark energy (DE) which can be thought of as a kind of space-filling fluid with constant energy density through the Universe [9, 49]. Another exotic form of matter proposed as a DE candidate is to consider a scalar field known as the quintessence [50]. On the other hand, the acceleration rate of the Universe in terms of the modified gravity theories continue to attract interest [49]. The simplest model which generalizes General Relativity is found by simply replacing the Ricci scalar (R) in the action by a function $f(R)$. This idea led to many modified gravity models studied in the literature [51] or in the cosmological set up of a higher-order modified teleparallel theory (see [52–59] and references therein). Instead of doing modification of gravity, the nonlinear electrodynamics can be used to avoid singularities as well as resolve the horizon problem [12].

Recently, the idea of nonlinear electrodynamics (NED) has been proposed as a solution to source the Universe acceleration [4–6]. In the early Universe the effect of the NED may have been very strong and, in principle, this may also explain the inflation. In this scenario the NEF can be considered as a source of the gravitational field and, as a consequence, nonlinear magnetic fields may be a driven mechanism of the inflation of the Universe. In this line of research, very recently many NED models have been investigated using a stochastic magnetic background with a non-vanishing $< B^2 >$, where matter should be identified with a primordial plasma [4, 5, 7, 11–14, 16], on the other hand, bulk viscosity term is neglected in the electric conductivity of the primordial plasma by taking $E^2 = 0$ [60–64]. The NED is useful to remove singularities of Big Bang and try to explain inflation naturally. Note that using the scalar fields for the inflation and early Universe have a problem with the fine exit. The problem is that after the inflation is started, it goes forever which is known as eternal inflation. However, we will propose a model where there is not any eternal inflation problem. For this purpose, we use the magnetic Universe with Nonlinear Electromagnetic Field (NEF) in the stochastic background with a nonzero value of $< B^2 >$ that supports the acceleration of the Universe. There are some cons and pros of NED from the Dirac-Born-Infeld (DBI) nonlinear electrodynamics such that for DBI theory, there is a duality symmetry, on the other hand, for the NED, it is broken but it is also violated for the QED. Other difference is the birefringence phenomenon that occur in NED and also in QED with quantum corrections, but not in DBI theory [14]. Moreover, the DBI model has a problem of causality.

In this paper we use a new model of Lagrangian of NED, dubbed “NED with an exponential correction”, which has a

Maxwell limit at low energies and which is different than DBI nonlinear electrodynamics [4]. We assume that radiation of NED is dominated in the early Universe, to solve the initial singularity problem. We show that the NED with the gravitation field can create the negative pressure and cause the inflation and cosmic acceleration of the cosmos. In this regard, our manuscript is extended version of the papers [4, 7, 14, 20, 21, 65], at which are used Nonlinear magnetic fields, as a source of inflation. This model of NED is valid for the early and current regime of the Universe, on the other hand for the late Universe, the magnetic field is very weak. However, the question whether one can explain the late regime of the Universe in terms of only NED remains, at least theoretically, as an open possibility. Hence, we additionally investigate whether the exponential NED can provide the late-time accelerated expansion of the Universe and then perform a phase-space analysis of Einstein-NED cosmology by including matter source [40–48, 66–76]. Using the phase-space on the cosmological principle, Einstein field equation easily becomes the dynamical system such as ODE (ordinary differential equation) naturally. The advantage of the using phase-space analysis is that one can do more stability analysis with using visual plots using the trajectories in geometrical way so that it becomes easy to observe the property with the help of the attractors which are the most easily seen experimentally [77]. On the other hand, conceptually using NED has the advantage that no need to use some exotic fields such as scalar fields, branes or extra dimensions, it is just photon fields, and it is well known also in nonlinear optics which studying behavior of light in nonlinear media and also nonlinear collision of particles in quantum electrodynamics [78–84].

The paper is organized as follows. In section II we introduce the Lagrangian of NED. In section III we examine the acceleration and evolution of the Universe in terms of NED fields. In section IV we perform a detailed phase-space analysis of our Einstein-NED cosmology model. In section V we consider a more realistic scenario, namely we include a matter source in our setup. In section VI we put constraints on the NED parameters, i.e. α and B_0, using observational Hubble data from cosmic chronometers and using the latest SNIa data. Finally we summarize and discuss our results in section VII.

II. GENERAL RELATIVITY COUPLED WITH NON-LINEAR ELECTRODYNAMICS WITH AN EXPONENTIAL CORRECTION

In highly nonlinear energy density situations such as in the early Universe, nonlinear electrodynamics is expected to play a crucial role in the evolution of the Universe [10, 19]. To reveal the mysteries of the cosmos, we first should understand the contributions of nonlinear fields to inflation. For this purpose, we propose the following action GR coupled with the NED field as follows:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R + \mathcal{L}_{NED}\right], \tag{1}$$

where M_{Pl} is the reduced Planck mass, R is the Ricci scalar and $\mathcal{L}_{NED}$ is the Lagrangian of the NED fields. The new NED Lagrangian density is chosen as follows:

$$\mathcal{L}_{NED} = -\frac{\mathcal{F}e^{-\alpha\mathcal{F}}}{(\alpha\mathcal{F} + \beta)}, \tag{2}$$

where α and β is dimensional parameters, $\mathcal{F} = (1/4)F_{\mu\nu}F^{\mu\nu} = (B^2 - E^2)/2$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. If we choose $\alpha \to 0$ and $\beta \to 1$, the Lagrangian reduces to the classical Maxwell's electrodynamics. Moreover, this Lagrangian does not appeal any exotic fields such as a scalar fields, extra dimensions, branes or modified Einstein fields.

After varying the action given in Eq. (1), one can find the Einstein field equations and the NED fields equation as follows:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa^2 T_{\mu\nu}, \tag{3}$$

where $\kappa^2 = 8\pi G$, $\kappa^{-1} = M_{Pl}$, and

$$\partial_\mu \left(\sqrt{-g}\frac{\partial\mathcal{L}_{NED}}{\partial\mathcal{F}}F^{\mu\nu}\right) = 0. \tag{4}$$

The energy momentum tensor [6]

$$T^{\mu\nu} = K_{\mu\lambda}F_\nu^\lambda - g^{\mu\nu}\mathcal{L}_{NED}, \tag{5}$$

with

$$K_{\mu\lambda} = \frac{\partial\mathcal{L}_{NED}}{\partial\mathcal{F}}F_{\mu\lambda}, \tag{6}$$

can be used to obtain the general form of the energy density ρ_{NED} and the pressure p_{NED} of NED fields by varying the action as following

$$\rho_{NED} = -\mathcal{L}_{NED} - E^2 \frac{\partial \mathcal{L}_{NED}}{\partial \mathcal{F}}, \tag{7}$$

$$= -\frac{\mathrm{e}^{-\alpha\,\mathcal{F}} \left(\left(E^2\alpha^2 - \alpha \right) \mathcal{F}^2 + \beta \, \left(E^2\alpha - 1 \right) \mathcal{F} - E^2\beta \right)}{\left(\alpha\,\mathcal{F} + \beta \right)^2}, \tag{8}$$

and

$$p_{NED} = \mathcal{L}_{NED} + \frac{\left(E^2 - 2B^2 \right)}{3} \frac{\partial \mathcal{L}_{NED}}{\partial \mathcal{F}} \tag{9}$$

$$= -\frac{2}{3} \frac{\left[\left(\left(B^2 - 1/2\,E^2 \right) \alpha^2 + 3/2\,\alpha \right) \mathcal{F}^2 + \beta \, \left(3/2 + \left(B^2 - 1/2\,E^2 \right) \right. \right.}{\left(\alpha\,\mathcal{F} + \beta \right)^2} \tag{10}$$

To find the solution of the Einstein field equations, we consider the homogeneous and isotropic cosmological metric of FRW with following line element:

$$ds^2 = -dt^2 + a(t)^2 \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right], \tag{11}$$

with scale factor $a(t)$.

The key point of this study is that it can be supposed that the stochastic magnetic fields are the cosmic background with the wavelength smaller than the curvature so we can use the averaging of EM fields which are sources in GR and then we can obtain the isotropic FRW spacetime [85]. The averaged

EM fields have the properties:

$$\langle E \rangle = \langle B \rangle = 0, \;\; \langle E_i B_j \rangle = 0, \tag{12}$$

$$\langle E_i E_j \rangle = \frac{1}{3} E^2 g_{ij}, \;\; \langle B_i B_j \rangle = \frac{1}{3} B^2 g_{ij},$$

where the averaging brackets $\langle \, \rangle$ is used for an average volume which is larger than the radiation wavelength and smaller than the curvature. The non-zero averaged magnetic field case is the most unexpected case [85]. Where the magnetic field of the Universe is frozen to occur the magnetic properties, it is necessary to screen the electric field of the charged primordial plasma. We use the Eqs. (8) and (10) (for $E^2 = 0$) and obtain:

$$\rho_{NED} = \frac{B^2 \mathrm{e}^{-1/2\,\alpha\,B^2}}{\alpha\,B^2 + 2\,\beta}, \tag{13}$$

and

$$p_{NED} = -\frac{2B^2 \left(B^4 \alpha^2 + 2\,\left(\beta + 3/4\right) \alpha\,B^2 - \beta\right) \mathrm{e}^{-\frac{\alpha B^2}{2}}}{3 \left(\alpha\,B^2 + 2\,\beta\right)^2}. \tag{14}$$

Afterwards, we use the FRW metric (11) and find the equation of the Friedmann:

$$3\frac{\ddot{a}}{a} = -\frac{\kappa^2}{2} \left(\rho_{NED} + 3p_{NED}\right). \tag{15}$$

Note that a dot "." over the letter means a time derivative. Furthermore, we check the condition of the accelerated Universe $\rho_{NED} + 3p_{NED} < 0$ with the sources of NED fields using into Eqs. (8) and (10):

$$\rho_{NED} + 3p_{NED} = -\frac{2B^2 \mathrm{e}^{-\frac{\alpha B^2}{2}} \left(B^4\alpha^2 + 2\,\beta\,\alpha\,B^2 + \alpha\,B^2 - 2\,\beta\right)}{\left(\alpha\,B^2 + 2\,\beta\right)^2}. \tag{16}$$

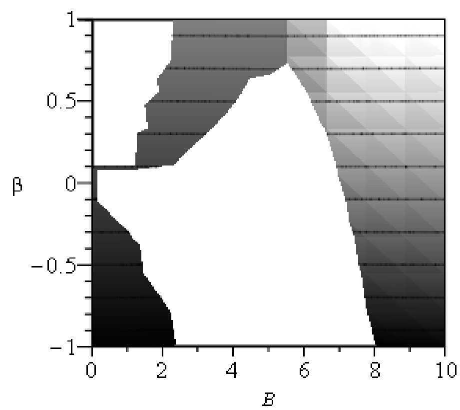

FIG. 1: The plots show the parameters of β and B which satisfy the acceleration of the Universe.

Universe accelerates ($\rho_{NED} + 3p_{NED} < 0$) as shown in Fig. (1) and the maximum acceleration occurs at $\alpha\,B^2 = -\frac{3}{2} +$

$\frac{1}{2}\sqrt{17}$ and maximum deceleration occurs at $\alpha\, B^2 = -\frac{3}{2} - \frac{1}{2}\sqrt{17}$ when $\beta = 1$. Note that the source of the strong nonlinear electrodynamics field accelerates the Universe in the early stages. Then we use the conservation of the energy-momentum tensor ($\nabla^\mu T_{\mu\nu} = 0$) for the FRW metric (11) and find the relation of fluid equation or known as also continuity equation as follows:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0. \tag{17}$$

It is noted that the Hubble parameter, or the Hubble constant is defined as $H = \frac{\dot{a}}{a}$ which is the expansion rate of our Universe. This equation gives

$$\frac{Be^{-\frac{1}{2}\alpha B^2}\left(2BH + \dot{B}\right)\left(-4\beta + \alpha^2 B^4 + 2\alpha\beta B^2\right)}{\left(2\beta + \alpha B^2\right)^2} = 0. \tag{18}$$

One can integrate (one of the branches of) the above equation [1] using the ρ and p to find the evolution of the magnetic field respect to the scale factor $B(t) = \frac{B_0}{a(t)^2}$ where B_0 is for $a(t) = 1$. Afterward, we rewrite the energy density ρ and the pressure p using the evolution of the magnetic field:

$$\rho_{NED} = \frac{{B_0}^2}{a^4} \mathrm{e}^{-\frac{\alpha\, {B_0}^2}{2a^4}} \left(\frac{\alpha\, {B_0}^2}{a^4} + 2\,\beta\right)^{-1}, \tag{19}$$

[1] We have also the trivial solution $B = 0$, the regime $\alpha B^2 \to \infty$ and the constant solutions B, such that $\left(-4\beta + \alpha^2 B^4 + 2\alpha\beta B^2\right) = 0$. We submit the reader to section IV A for a more complete discussion of this special cases.

$$p_{NED} = -\frac{2{B_0}^2}{3a^4} e^{-\frac{\alpha\,{B_0}^2}{2a^4}} \left(\frac{\alpha\,{B_0}^2}{a^4} + 2\,\beta\right)^{-2} \times$$
$$\left(\frac{{B_0}^4\alpha^2}{a^8} + 2\,\frac{\alpha\,(\beta+3/4)\,{B_0}^2}{a^4} - \beta\right). \tag{20}$$

When we check the limits of the energy density and pressure, we conclude that there is no singularity point at $a(t) \to 0$ and $a(t) \to \infty$,

$$\lim_{a(t)\to 0} \rho(t) = \lim_{a(t)\to 0} p(t) = \lim_{a(t)\to\infty} \rho(t) = \lim_{a(t)\to\infty} p(t) = 0. \tag{21}$$

Using the equation of state (EoS)

$$\omega = \frac{p(t)}{\rho(t)} = -\frac{2}{3}\left(\frac{B_0^4\alpha^2}{a^8} + 2\,\frac{\alpha\,(\beta+3/4)\,B_0^2}{a^4} - \beta\right) \times$$
$$\left(\frac{\alpha\,B_0^2}{a^4} + 2\,\beta\right)^{-1}, \tag{22}$$

then we use Eq.s (19 and 20) and obtain the radiation or other relativistic fluid case:

$$\lim_{a\to\infty} \omega = \frac{1}{3}. \tag{23}$$

Now, at $Y = \frac{\alpha B_0^2}{a^4} = \left(\sqrt{5}-1\right)$, we have EoS corresponding to de Sitter spacetime $\omega = -1$ (the usual cosmological constant), at $Y = \frac{65-\sqrt{7}}{4}$, we have $\omega = 0$ for non-relativistic matter (baryons, CDM), also for some values we have $-1 < \omega < -1/3$ for "quintessence", dynamic dark energy resulting in ultimate acceleration of the universal expansion as shown in Fig. 2 plotted for w versus $Y = [\alpha B_0^2/a^4]$. We will not consider $\omega < -1$ which has been termed "phantom" dark energy. We can also refer dark energy as vacuum energy, because one possible source of dark energy is quantum-mechanical fluctuations of the vacuum.

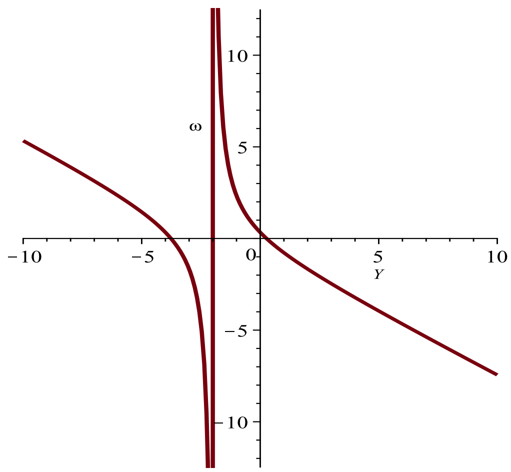

FIG. 2: The plot shows the relation between ω versus Y.

The Ricci scalar, which represents the curvature of space-time, is calculated by using Einstein's field equation (3) and the energy-momentum tensor as follows:

$$R = \kappa^2(\rho - 3p) = 2\,\frac{\kappa^2 {B_0}^4 \alpha\,\left(2\,\beta\,a^4 + 2\,a^4 + \alpha\,{B_0}^2\right)}{a^4\left(2\,\beta\,a^4 + \alpha\,{B_0}^2\right)^2} \mathrm{e}^{-\frac{\alpha {B_0}^2}{2a^4}}, \tag{24}$$

$$\lim_{a(t)\to\infty} R(t) = \lim_{a(t)\to 0} R(t) = 0.$$

The Ricci tensor squared $R_{\mu\nu}R^{\mu\nu}$ and the Kretschmann scalar

$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ are also obtained as

$$R_{\mu\nu}R^{\mu\nu} = \kappa^4\left(\rho^2 + 3p^2\right), \tag{25}$$

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \kappa^4\left(\frac{5}{3}\rho^2 + 2\rho p + 3p^2\right). \tag{26}$$

To show the non-singular curvature, Ricci tensor, and the Kretschmann scalar, the effect of the scale factor is investigated by taking the limit of Eq. (24) for the Universe accelerates at $a(t) \to 0$ and at $a(t) \to \infty$ according to Eq. (21). We show that the spacetime will be flat at $t \to \infty$ and singularities is removed at the early/late phase of the Universe.

III. ACCELERATION AND EVOLUTION OF THE UNIVERSE

Now, we find the evolution of the Universe using the Einstein's equations and the energy density given in Eq.s (19 and 20). Without considering dust like matter, we study the evolution of the Universe. To find the scale factor as a function of time, first we use the second Friedmann's equation in flat Universe:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho_{NED}}{3}, \tag{27}$$

and one can find the equation which shows conservation of energy for a particle moving in a effective potential $V(a)$:

$$\dot{a}^2 + V_{\text{eff}}(a) = 0, \tag{28}$$

with $V_{\text{eff}}(a) = -\frac{1}{6}\,\frac{{B_0}^2}{a^2}\text{e}^{-\frac{\alpha\,{B_0}^2}{2a^4}}\left(\frac{\alpha\,{B_0}^2}{2a^4} + 1\right)^{-1}$.

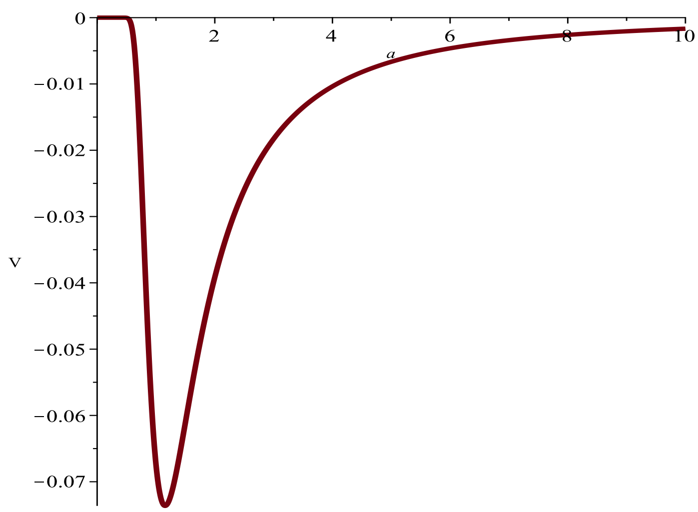

FIG. 3: The function of effective potential $V_{\text{eff}}(a)$ vs. scale factor a.

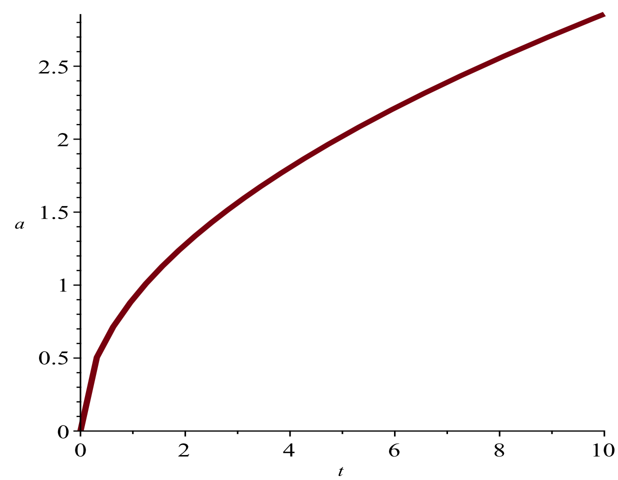

FIG. 4: The function of scale factor $a(t)$ vs. cosmic time t.

We use equation (28), to get a qualitative feel for the evolution of the early Universe. The plot of the effective potential $V_{\text{eff}}(a)$ shows us the behavior of the scale factor in the early Universe. The effective potential with a positive slope yields a force tending to slow down positive motion along the horizontal axis, while the portion of the effective potential with a negative slope yields a force tending to speed up positive motion along the horizontal axis. These two conditions occur to the right and the left, respectively, of the minimum point at $a(t) \approx 1.15$. By analogy, then, $a(t)$ accelerates to the left of $a(t) \approx 1.15$ and decelerates to the right of $a(t) \approx 1.15$. This acceleration is due to nonlinear electrodynamics which behavior similarly to dark energy. We calculate the value of the critical scale factor as $a_c = \frac{\sqrt{2}\sqrt[4]{\left(2\,\beta+1+\sqrt{4\,\beta^2+12\,\beta+1}\right)B_0{}^2\alpha\,\beta^3}}{2\beta}$ using the Eq. (15). Using the Eq. (27) with the energy density in Eq. (19), we obtain the equation as follows:

$$\dot{a}^2 = \frac{1}{3}\frac{B_0{}^2a^2}{2\,\beta\,a^4+\alpha\,B_0{}^2}\mathrm{e}^{-\frac{\alpha\,B_0{}^2}{2a^4}}. \tag{29}$$

After simplify the Eq. (29) by using asymptotically expansion of B_0 and integrate, we calculate approximate cosmic time [86] as follows:

$$t = \frac{\sqrt{6}}{2}\frac{a^2}{B_0} + t_0, \tag{30}$$

where t_0 is a constant of integration which gives only the shift in time as shown in Fig. (4. Note that the integral constant is $t_0 = 0$. One can also find the evaluation of the scale factor for $(\alpha = B_0 = 1)$ as follows:

$$a = \sqrt[4]{2/3}\sqrt{t}. \tag{31}$$

Note that evolution of the scale factor has similar feature with radiation dominated Universe and for $t = 0$ it reduces to zero

$$a_0 = a(t = 0) = 0. \tag{32}$$

The function of a_0 is a radius of the Universe which shows that Universe begins from the zero point. However if one consider the integration constant of t_0, it can be shifted. One can also show them in terms of redshift where $a = (1 + z)^{-1}$.

The next step is to check the causality of the Universe using the speed of the sound. The speed of the sound should be smaller than the local light speed ($c_s \leq 1$) [87]. Moreover, we check the square sound speed that must be positive ($c_s^2 > 0$). If the Universe satisfied these conditions, there is a classical stability. The square of the sound speed is found from Eq.s (8) and (10);

$$c_s^2 = \frac{dp}{d\rho} = \frac{dp/d\mathcal{F}}{d\rho/d\mathcal{F}} = \frac{-24\,a^{16}\beta^2 - 20\,\beta\,\left(\beta + \frac{11}{5}\right)\alpha\,{B_0}^2 a^{12} + 8\,(\beta -}{24\,a^{16}\beta^2 - 12\,{B_0}^2\beta\,\alpha\,(\beta - 1} \tag{33}$$

and the classical stability ($c_s^2 > 0$) and causality of the Universe using the speed of sound $c_s \leq 1$, occurs as shown in Fig. 5 and Fig. 6, respectively.

IV. PHASE SPACE ANALYSIS

The equation (28) represents the motion of a particle of the unit mass in the potential $U_{\text{eff}}(a) = -\frac{\rho_{NED}(a)a^2}{6}$. We have that the equation (28) is satisfied on the zero energy level, where ρ_{NED} plays the role of effective energy density parameterized

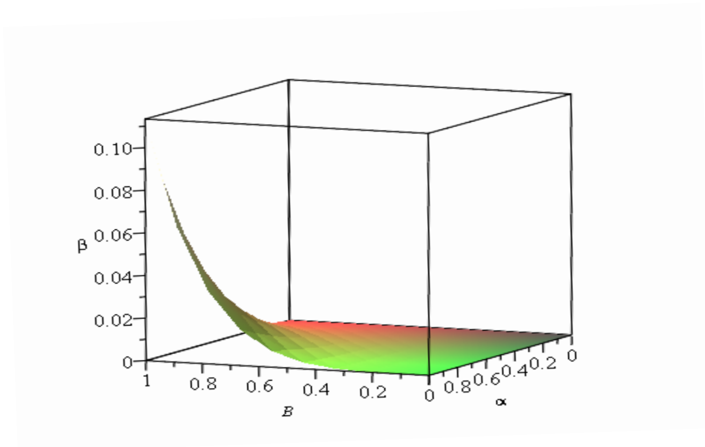

FIG. 5: The plot shows the classical stability with variables α, β and B.

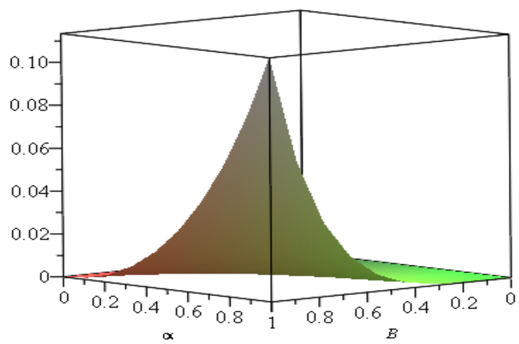

FIG. 6: The plot shows the causality with variables α, β and B.

through the scale factor $a(t)$. Therefore the standard cosmological model can be simply represented in the terms of a dynamical system of a Newtonian type:

$$\ddot{a} = -\frac{\partial U_{\text{eff}}}{\partial a}, \quad U_{\text{eff}} = -\frac{B_0^2 a^2 e^{-\frac{\alpha B_0^2}{2a^2}}}{6\left(2\beta a^2 + \alpha B_0^2\right)}, \tag{34}$$

where the scale factor a plays the role of a positional variable of a fictitious particle of the unit mass, which mimics the expansion of the Universe.

For simplicity, we use introduce the new constant $\alpha_0 = \frac{1}{2}{B_0}^2\alpha$, and introduce the time rescaling $\tau = \frac{t}{B_0}, B_0 > 0$, and the variables

$$x = a, \quad y = \frac{\dot{a}}{B_0}, \tag{35}$$

The system is then equivalent to

$$\frac{dx}{d\tau} = y, \quad \frac{dy}{d\tau} = \frac{\alpha_0 e^{-\frac{\alpha_0}{x^2}} \left(\alpha_0 + (\beta+1)x^2\right)}{6x\left(\alpha_0 + \beta x^2\right)^2}. \tag{36}$$

By definition $x \geq 0$ (since we consider that the scale factor is non-negative). For this reason, it is convenient to define the new variables

$$x = e^u, y = v, \tag{37}$$

which takes values on the real line, and the time variable

$$\eta = \int a^{-1} d\tau \equiv \int e^{-u} d\tau. \tag{38}$$

This system can be written in the form

$$\frac{du}{d\eta} = v, \quad \frac{dv}{d\eta} = -\frac{\partial W(u)}{\partial u}. \tag{39}$$

Thus, $\frac{v^2}{2} + W(u) = E$, is the constant of energy. From the above system we see that, generically, the fixed points are situated on the axis u $(v = 0)$. From the characteristic equation it follows that just three types of fixed points are admitted:

1. saddle if $u_0 : \frac{\partial W}{\partial u}|_{u=u_0} = 0$ and $\frac{\partial^2 W}{\partial u^2}|_{u=u_0} < 0$;
2. focus if $\frac{\partial^2 W}{\partial u^2}|_{u=u_0} > 0$;
3. degenerated critical point if $\frac{\partial^2 W}{\partial u^2}|_{u=u_0} = 0$.

In the concrete example we have

$$W(u) = -\frac{e^{-\alpha_0 e^{-2u}}}{12\,(\beta + \alpha_0 e^{-2u})}. \tag{40}$$

Hence, the fixed point at the finite region if the phase space is $A : (u, v) = \left(\ln\left(\sqrt{\frac{\alpha_0}{-\beta-1}}\right), 0\right)$, that exists for $\alpha_0 > 0, \beta < -1$, or $\alpha_0 < 0, \beta > -1$. The eigenvalues are $\left\{-\frac{ie^{\frac{\beta+1}{2}}(\beta+1)}{\sqrt{3}}, \frac{ie^{\frac{\beta+1}{2}}(\beta+1)}{\sqrt{3}}\right\}$.

For $\alpha_0 > 0, \beta < -1$, or $\alpha_0 < 0, \beta > -1$, $U''(x_0) = \frac{1}{3}e^{\beta+1}(\beta+1)^2 > 0$ (where x_0 is the coordinate of A). So that, according to the previous classification they are focus.
Now, in the limit $u \to +\infty$, we have $W'(u) \to 0, W''(u) \to 0$, hence, in this regime we can expect to have degenerate critical points. Furthermore, for $\frac{\alpha_0}{\beta+1} > 0$ there are no fixed points at the finite region of the phase-plane.

Since the above system is in general unbounded, then we introduce the compactification

$$U = \frac{u}{\sqrt{1+u^2+v^2}}, \quad V = \frac{v}{\sqrt{1+u^2+v^2}}, \tag{41}$$

we have the equivalent flow

$$\frac{dU}{d\eta} = -\frac{\alpha_0 e^{-\alpha_0 e^{-\frac{2U}{\sqrt{1-U^2-V^2}}}} UV\sqrt{1-U^2-V^2}\left(\alpha_0 + (\beta+1)e^{\frac{2U}{\sqrt{1-U^2-}}}\right.}{6\left(\alpha_0 + \beta e^{\frac{2U}{\sqrt{1-U^2-V^2}}}\right)^2} \tag{42}$$

$$\frac{dV}{d\eta} = \frac{\alpha_0 e^{-\alpha_0 e^{-\frac{2U}{\sqrt{1-U^2-V^2}}}} \left(1-V^2\right)\sqrt{1-U^2-V^2}\left(\alpha_0 + (\beta+1)e^{\sqrt{1}}\right.}{6\left(\alpha_0 + \beta e^{\frac{2U}{\sqrt{1-U^2-V^2}}}\right)^2} \tag{43}$$

For the choice $\alpha_0 = 1, \beta = -2$, the equations near the origin of coordinates (fixed point A), can be written as

$$\frac{dU}{d\eta} = V, \quad \frac{dV}{d\eta} = -\frac{U}{3e}. \tag{44}$$

where we have neglected the higher order terms, such that, close to the origin, the solutions can be approximated by

$$U(\eta) = U_0 \cos\left(\frac{\eta}{\sqrt{3e}}\right) + \sqrt{3e}V_0 \sin\left(\frac{\eta}{\sqrt{3e}}\right), \tag{45}$$

$$V(\eta) = V_0 \cos\left(\frac{\eta}{\sqrt{3e}}\right) - \frac{U_0 \sin\left(\frac{\eta}{\sqrt{3e}}\right)}{\sqrt{3e}}. \tag{46}$$

Thus, the orbits of the original system near the origin can be approximated by the ellipses given by

$$U(\eta)^2 + 3eV(\eta)^2 = \text{constant} = U_0^2 + 3eV_0^2, \tag{47}$$

which leads to periodic solutions.

Using the above approximation we get the parametric solution

$$a = \exp\left(\frac{\sqrt{3e}c_2 \sin\left(\frac{\eta}{\sqrt{3e}}\right) + c_1 \cos\left(\frac{\eta}{\sqrt{3e}}\right)}{\sqrt{1 - \left(c_2 \cos\left(\frac{\eta}{\sqrt{3e}}\right) - \frac{c_1 \sin\left(\frac{\eta}{\sqrt{3e}}\right)}{\sqrt{3e}}\right)^2 - \left(\sqrt{3e}c_2 \sin\left(\frac{\eta}{\sqrt{3e}}\right) \right.}} \right. \tag{48}$$

$$\dot{a} = \frac{B_0 \left(c_2 \cos\left(\frac{\eta}{\sqrt{3e}}\right) - \frac{c_1 \sin\left(\frac{\eta}{\sqrt{3e}}\right)}{\sqrt{3e}}\right)}{\sqrt{1 - \left(c_2 \cos\left(\frac{\eta}{\sqrt{3e}}\right) - \frac{c_1 \sin\left(\frac{\eta}{\sqrt{3e}}\right)}{\sqrt{3e}}\right)^2 - \left(\sqrt{3e}c_2 \sin\left(\frac{\eta}{\sqrt{3e}}\right) + c_1 \right.}} \tag{49}$$

The relation between η and the cosmic time t can be found by using

$$t - t_0 = B_0 \int_0^\eta a(\vartheta) d\vartheta, \tag{50}$$

which in general must be integrated numerically.

In the figure 7, we show the dynamics of system (42)-(43) for $\alpha_0 = 1$ and $\beta = -2$ (left panel) and $\alpha_0 = 1$ and $\beta = 1$ (right panel). In the left panel there is only a fixed point at the finite region that coincides with the origin (for this choice of parameters). In the right panel there are not fixed points at the finite region of the phase space.

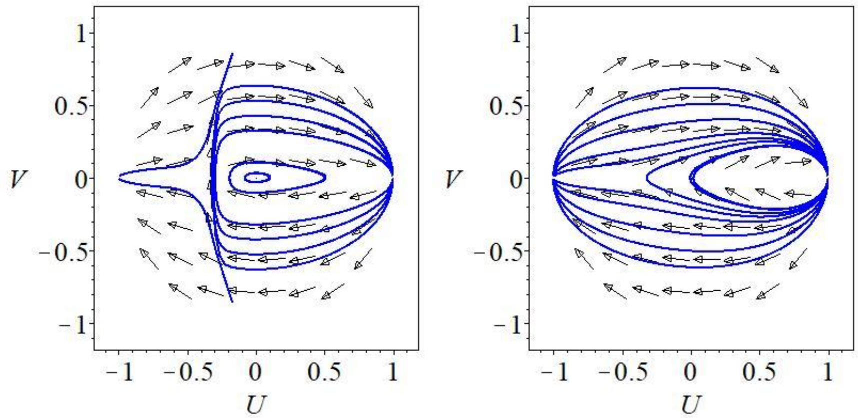

FIG. 7: Dynamics of system (42)-(43) for $\alpha_0 = 1$ and $\beta = -2$ (left panel) and $\alpha_0 = 1$ and $\beta = 1$ (right panel).

In the figure 8 it is shown the behavior of the scale factor and its first derivative in terms of the parameter η. This is the realization of a cyclic Universe in our model supported by NED. The scale factor goes below and above the value $a = 1$, reaching a maximum and a minimum value of a, and a is bounded away zero (there is no initial singularity, as expected from our NED proposal).

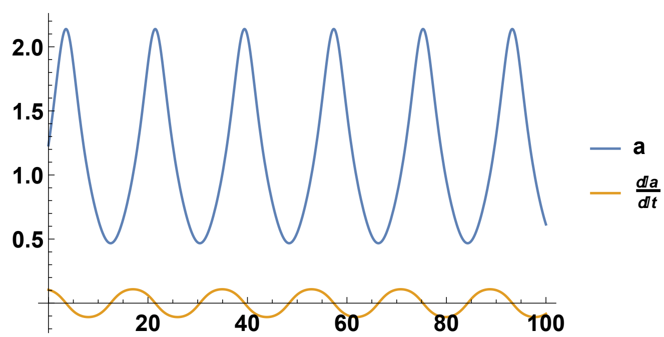

FIG. 8: In the figure it is shown a typical behavior in the case $\alpha_0 = 1, \beta = -2$ (we have chosen for the numerics $B_0 = 0.5, c_1 = c_2 = 0.2$).

For arbitrary α_0 and β, we obtain, by expanding up to second order around A, and neglecting higher orders terms

$$\frac{dU}{d\eta} = V\left(-\frac{1}{12}e^{\beta+1}(\beta+1)^2\ln^2(s) - \frac{4}{\ln^2(s)+4} + 2\right) \quad (51)$$

$$\frac{dV}{d\eta} = -\frac{1}{12}e^{\beta+1}(\beta+1)^2\ln(s)\sqrt{\ln^2(s)+4}$$
$$-\frac{1}{12}e^{\beta+1}(\beta+1)^2 U\left(\ln^2(s)+4\right). \quad (52)$$

where we have defined by simplicity of notation $s = -\frac{\beta+1}{\alpha_0}$ and assumed $s \geq 0$. The solution is given by

$$U(\eta) = -\frac{\ln(s)}{\sqrt{\ln^2(s)+4}} + c_1\cosh\left(\frac{1}{12}e^{\frac{\beta+1}{2}}(\beta+1)\Delta\eta\right) - \frac{c_2\Delta e^{-\frac{1}{12}(\beta+}}{} \quad (53)$$

$$V(\eta) = c_2\cosh\left(\frac{1}{12}e^{\frac{\beta+1}{2}}(\beta+1)\Delta\eta\right) + -\frac{(\beta+1)c_1\left(\ln^2(s)+4\right)e^{-\frac{1}{12}}}{} \quad (54)$$

where $\Delta = \sqrt{\ln^2(s)\left(e^{\beta+1}(\beta+1)^2\left(\ln^2(s)+4\right)-24\right)-48}$. The solutions are periodic for $\ln^2(s)\left(e^{\beta+1}(\beta+1)^2\left(\ln^2(s)+4\right)-24\right)$ $48 < 0$. This solution approximates the exact solutions of the full system surrounding A. Once we know the expressions of $U(\eta), V(\eta)$ we can calculate the parametric expressions of a

and $\dot{a}$ as functions of η through

$$a = e^{\frac{U}{\sqrt{1-U^2-V^2}}}, \quad \dot{a} = \frac{B_0 V}{\sqrt{1-U^2-V^2}}, \tag{55}$$

where U and V are defined by (53) and (54), respectively. Finally, the dynamics at the circle at infinity can be represented by the flow of

$$\frac{dU}{d\eta} = V^3, \quad \frac{dV}{d\eta} = -UV^2. \tag{56}$$

The orbits lying on the circle at infinity can be parametrized as

$$U(\eta) = \pm\frac{c_2+\eta}{\sqrt{(c_2+\eta)^2+1}}, \quad V(\eta) = \sqrt{\frac{1}{(c_2+\eta)^2+1}}$$

or

$$U(\eta) = \pm\frac{\eta-c_2}{\sqrt{(\eta-c_2)^2+1}}, \quad V(\eta) = -\sqrt{\frac{1}{(\eta-c_2)^2+1}}.$$

Thus, there are fixed points lying on the circumference at infinity $(U, V) = (\pm 1, 0)$. Thus, apart of the singular points at infinity described before, and the point at the finite region $(U, V) = \left(\frac{\ln\left(-\frac{\beta+1}{\alpha_0}\right)}{\sqrt{\ln^2\left(-\frac{\beta+1}{\alpha_0}\right)+4}}, 0\right)$, we have that for $\alpha_0\beta < 0$, there exists the singular line:

$$\left\{(X, Y) : \alpha_0 + \beta e^{\frac{2U}{\sqrt{1-U^2-V^2}}} = 0\right\}. \tag{57}$$

For $\alpha_0 < 0, \beta > 0$ or $\alpha_0 > 0, \beta < -1$ the point A and this line exist.

A. Integrability and connection with the observables

In this section we comment on the integrability of the system at hand, and calculate some observables in terms of redshift.

As we commented before, from the conservation of the energy-momentum tensor ($\nabla^\mu T_{\mu\nu} = 0$) for the FRW metric we have

$$\frac{Be^{-\frac{1}{2}\alpha B^2}\left(2BH+\dot{B}\right)\left(-4\beta+\alpha^2 B^4+2\alpha\beta B^2\right)}{\left(2\beta+\alpha B^2\right)^2}=0, \quad (58)$$

and from (13) we have

$$3H^2 := \rho_{NED} = \frac{B^2 \mathrm{e}^{-1/2\,\alpha\, B^2}}{\alpha\, B^2 + 2\,\beta}. \quad (59)$$

As we mentioned before, we have the trivial solution $B = 0$, the regime $\alpha B^2 \to \infty$ and the constant solutions B, such that $\left(-4\beta+\alpha^2 B^4+2\alpha\beta B^2\right) = 0$. Concerning the former solutions we have the following results:

1. $\alpha B^2 = -\beta - \sqrt{\beta^2 + 4\beta}$, is a real value for $\beta > 0, \alpha < 0$ or $\beta \leq -4, \alpha > 0$. Equation (59) means that H is constant and equal to a cosmological constant $H = \sqrt{\Lambda}$, with

$$\Lambda = \frac{e^{\frac{1}{2}\left(\beta+\sqrt{\beta(\beta+4)}\right)}\left(\beta+\sqrt{\beta(\beta+4)}+2\right)}{6\alpha},$$

but under the above conditions results in $\Lambda < 0$, which implies $a \propto e^{i\sqrt{|\Lambda|}t}$. So, we discard these solutions. Furthermore, we have a second class of solutions given by

2. $\alpha B^2 = -\beta + \sqrt{\beta^2 + 4\beta}$ is a real value for $\beta > 0, \alpha > 0$ or $\beta \leq -4, \alpha > 0$. As before, equation (59) means that H is constant and equal to a cosmological constant $H = \sqrt{\Lambda}$ with

$$\Lambda = \frac{e^{\frac{1}{2}\left(\beta - \sqrt{\beta(\beta+4)}\right)}\left(\beta - \sqrt{\beta(\beta+4)} + 2\right)}{6\alpha}.$$

For $\beta \leq -4, \alpha > 0$ we have $\Lambda < 0$, which implies $a \propto e^{i\sqrt{|\Lambda|}t}$. So, we discard these solutions. However, for $\beta > 0, \alpha > 0$, we obtain $\Lambda > 0$ as required in, and then we arrive a a regime similar to de-Sitter type Universe, with $a \propto e^{\sqrt{\Lambda}t}$.

Summarizing, our model supports the inflation similar to de-Sitter Universe, with

$$a \propto e^{\sqrt{\Lambda}t}, \quad \Lambda = \frac{e^{\frac{1}{2}\left(\beta - \sqrt{\beta(\beta+4)}\right)}\left(\beta - \sqrt{\beta(\beta+4)} + 2\right)}{6\alpha}, \tag{60}$$

provided $\beta > 0, \alpha > 0$.

Now, assuming that B is not a constant or trivial, so, it is easy to recast the field equations as

$$\dot{B} = -2BH, \quad \dot{H} = \frac{B^2 e^{-\frac{1}{2}\alpha B^2}\left(-4\beta + \alpha^2 B^4 + 2\alpha\beta B^2\right)}{3\left(2\beta + \alpha B^2\right)^2}, \tag{61}$$

The other restriction from Eq. (59) is

$$\frac{B^2 e^{-\frac{1}{2}\alpha B^2}}{2\beta + \alpha B^2} = 3H^2 \geq 0. \tag{62}$$

Using the previous restriction and introducing the logarithmic time variable $N = \ln a$, we obtain the equations

$$\frac{dB}{dN} = -2B, \quad \frac{dH}{dN} = \frac{H\left(-4\beta + \alpha^2 B^4 + 2\alpha\beta B^2\right)}{2\beta + \alpha B^2}, \tag{63}$$

which are integrable and lead to

$$B(a) = \frac{c_1}{a^2}, \quad H(a) = \frac{c_2 e^{-\frac{\alpha c_1^2}{4a^4}}}{a^2\sqrt{\frac{\alpha c_1^2}{a^4} + 2\beta}}, \tag{64}$$

which satisfy the compatibility condition

$$\frac{\alpha\left(c_1^2 - 3c_2^2\right)e^{-\frac{\alpha c_1^2}{2a^4}}}{3\left(2\beta a^4 + \alpha c_1^2\right)} = 0, \tag{65}$$

which is satisfied for all α, β, and a, if $c_1^2 - 3c_2^2 = 0$. The physical conditions $B \geq 0, H \geq 0$ (required for an expanding

Universe) implies $c_1 \geq 0, c_2 \geq 0$. Hence, $c_2 = c_1/\sqrt{3}$. Thus,

$$H(a) = \frac{c_1 e^{-\frac{\alpha c_1^2}{4a^4}}}{a^2\sqrt{\frac{3\alpha c_1^2}{a^4} + 6\beta}}. \tag{66}$$

The deceleration parameter is given by

$$q := -1 - \frac{\dot{H}}{H^2} = -1 - a\frac{d\ln H}{da} \tag{67}$$

$$= -1 + \frac{4\beta a^4}{2\beta a^4 + \alpha c_1^2} - \frac{\alpha c_1^2}{a^4}. \tag{68}$$

In terms of redshift we have

$$B(z) = c_1(z+1)^2, \tag{69}$$

$$H(z) = \frac{c_1(z+1)^2 e^{-\frac{1}{4}\alpha c_1^2 (z+1)^4}}{\sqrt{6\beta + 3\alpha c_1^2 (z+1)^4}}, \tag{70}$$

$$q = -1 + \frac{4\beta}{2\beta + \alpha c_1^2(z+1)^4} - \alpha c_1^2 (z+1)^4, \tag{71}$$

where we have taken the reference values $z = 0, a = 1$ for today, such that, as $z \to -1$, the contribution of magnetic fields at cosmological scales is negligible [88]. Defining $B(0) = B_0, H(0) = H_0$, the current values of B, and H, we obtain the relations for the parameters

$$\beta = \frac{1}{6}{B_0}^2\left(\frac{e^{-\frac{1}{2}\alpha {B_0}^2}}{{H_0}^2} - 3\alpha\right). \tag{72}$$

Henceforth,

$$H(z) = \frac{(z+1)^2 e^{-\frac{1}{4}\alpha {B_0}^2 (z+1)^4}}{\sqrt{-3\alpha + \frac{e^{-\frac{1}{2}\alpha {B_0}^2}}{{H_0}^2} + 3\alpha(z+1)^4}}. \tag{73}$$

For the deceleration factor we have

$$q = -1 - \alpha {B_0}^2 (z+1)^4 + \frac{2 - 6\alpha {H_0}^2 e^{\frac{1}{2}\alpha {B_0}^2}}{3\alpha {H_0}^2 z(z+2)(z(z+2)+2) e^{\frac{1}{2}\alpha {B_0}^2} + 1}, \tag{74}$$

with current value

$$q_0 = 1 - \alpha \left(6{H_0}^2 e^{\frac{1}{2}\alpha {B_0}^2} + {B_0}^2 \right) \tag{75}$$

$$\tag{76}$$

Thus, to accommodate the current accelerated phase it is required $\alpha > \left(6{H_0}^2 e^{\frac{1}{2}\alpha {B_0}^2} + {B_0}^2 \right)^{-1}$.

V. INCLUDING MATTER

To introduce a more realistic model we include an additional matter source with constant equation of state parameter $w_m = p_m/\rho_m$ and with continuity equation

$$\dot{\rho_m} + 3H(1 + w_m)\rho_m = 0. \tag{77}$$

Integrating out the last equation we find $\rho_m = \rho_{m,0} a^{-3(1+w_m)}$. In this case, the second Friedmann's equation in flat Universe becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho_{NED} + \rho_{m,0} a^{-3(1+w_m)}}{3}, \tag{78}$$

Hence, the effective potential is now

$$U_{\text{eff}} = -\frac{B_0^2 a^2 e^{-\frac{\alpha B_0^2}{2a^2}}}{6\left(2\beta a^2 + \alpha B_0^2\right)} - \frac{\rho_{m,0}}{6} a^{-1-3w_m}. \tag{79}$$

As in the previous case, we introduce the new constants $\alpha_0 = \frac{1}{2} {B_0}^2 \alpha$ and $\rho_{m,0} = \rho_0 B_0^2$, the variables

$$a = e^u, \frac{\dot{a}}{B_0} = v, \tag{80}$$

which takes values on the real line, and the new time variable

$$\eta = \frac{1}{B_0} \int a^{-1} dt \equiv \frac{1}{B_0} \int e^{-u} dt, \tag{81}$$

The system is then equivalent to

$$\frac{du}{d\eta} = v, \tag{82}$$

$$\frac{dv}{d\eta} = \frac{\alpha_0 e^{-\alpha_0 e^{-2u}} \left(\alpha_0 + (\beta+1) e^{2u}\right)}{6\left(\alpha_0 + \beta e^{2u}\right)^2} - \frac{1}{6} \rho_0 (3w_m + 1) e^{-u(3w_m+1)}. \tag{83}$$

Now, in this model the effective potential is

$$U_{\text{eff}}(u) = -\frac{1}{6}\rho_0 e^{-u(3w_m+1)} - \frac{e^{-\alpha_0 e^{-2u}}}{12\left(\beta + \alpha_0 e^{-2u}\right)}. \tag{84}$$

That is,

$$\frac{du}{d\eta} = v, \quad \frac{dv}{d\eta} = -\frac{\partial U_{\text{eff}}}{\partial u}. \tag{85}$$

Now, the fixed points are found by solving numerically

$$\frac{\alpha_0 e^{-\alpha_0 e^{-2u}}\left(\alpha_0 + (\beta+1)e^{2u}\right)}{6\left(\alpha_0 + \beta e^{2u}\right)^2} - \frac{1}{6}\rho_0(3w_m+1)e^{-u(3w_m+1)} = 0. \tag{86}$$

As before the above system is in general unbounded, so that we introduce the compactification

$$U = \frac{u}{\sqrt{1+u^2+v^2}}, \quad V = \frac{v}{\sqrt{1+u^2+v^2}}. \tag{87}$$

Hence, we have the equivalent flow

$$\frac{dU}{d\eta} = -\frac{\alpha_0 e^{-\alpha_0 e^{-\frac{2U}{\sqrt{1-U^2-V^2}}}} \left(\alpha_0 + (\beta+1) e^{\frac{2U}{\sqrt{1-U^2-V^2}}}\right) UV\sqrt{1-U^2 -}}{6\left(\alpha_0 + \beta e^{\frac{2U}{\sqrt{1-U^2-V^2}}}\right)^2}$$
$$+ \frac{1}{6}\rho_0 UV(3w_m+1)\sqrt{1-U^2-V^2} e^{-\frac{U(3w_m+1)}{\sqrt{1-U^2-V^2}}} + V(1-U^2-V^2) \quad (88)$$

$$\frac{dV}{d\eta} = -\frac{\alpha_0 e^{-\alpha_0 e^{-\frac{2U}{\sqrt{1-U^2-V^2}}}} \left(\alpha_0 + (\beta+1) e^{\frac{2U}{\sqrt{1-U^2-V^2}}}\right) \left(V^2-1\right)\sqrt{1 -}}{6\left(\alpha_0 + \beta e^{\frac{2U}{\sqrt{1-U^2-V^2}}}\right)^2}$$
$$+ \frac{1}{6}\rho_0 e^{-\frac{U(3w_m+1)}{\sqrt{1-U^2-V^2}}} \left(V^2-1\right)(3w_m+1)\sqrt{1-U^2-V^2} - UV^2. \quad (89)$$

Now, we proceed to the numerical integration of system (88)-(89) for some values of α_0 and β, and for a) dust (and the results are presented in Fig. 9) and b) for stiff matter (and the results are presented in Fig. 10).

As in the model without matter, there is a numerical evidence of the realization of a cyclic Universe in our model supported by NED as shown in the left panels of Figs. 9) and 10 respectively. That is, the scale factor goes below and above the value $a = 1$, reaching a maximum and a minimum value of a, and a is bounded away zero, as expected from our NED proposal. Indeed, if we consider the special choice of

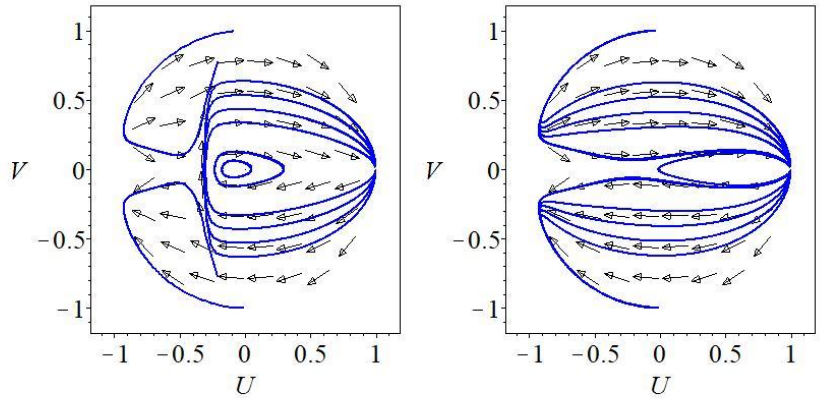

FIG. 9: Dynamics of system (88)-(89) for the values $\alpha_0 = 1, \beta = -2, \rho_0 = 0.1, w_m = 0$ (left panel) and $\alpha_0 = 1, \beta = 2, \rho_0 = 0.1, w_m = 0$ (right panel).

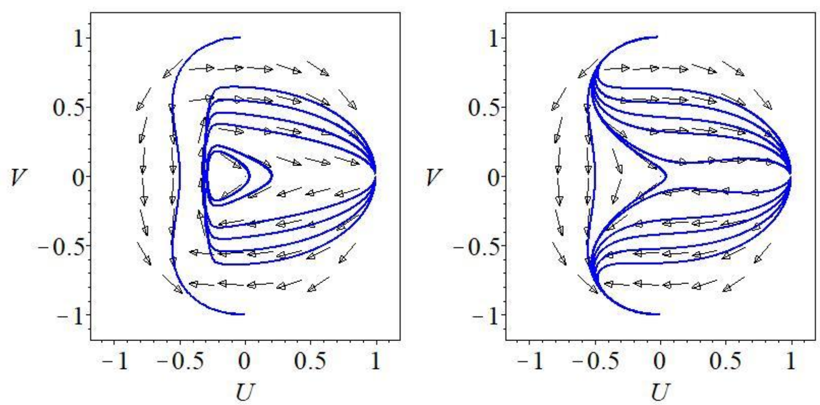

FIG. 10: Dynamics of system (88)-(89) for the values $\alpha_0 = 1, \beta = -2, \rho_0 = 0.1, w_m = 1$ (left panel) and $\alpha_0 = 1, \beta = 2, \rho_0 = 0.1, w_m = 1$ (right panel)

parameters

$$\frac{e^{-\alpha_0}\alpha_0(\alpha_0+\beta+1)}{6(\alpha_0+\beta)^2} - \frac{1}{6}\rho_0(3w_m+1) = 0, \tag{90}$$

then it follows that the origin is a fixed point of the dynamical system. By taking a linear expansion around the origin, we find the approximate system

$$\frac{dU}{d\eta} = V, \quad \frac{dV}{d\eta} = -\gamma U, \tag{91}$$

$$\gamma = -\frac{e^{-\alpha_0}\alpha_0\left(2\alpha_0^3 + \alpha_0^2(4\beta + 3w_m + 3) + \alpha_0(2\beta(\beta+1) + (6\beta+3)\cdots\right.}{6(\alpha_0+\beta)^3 \cdots} \tag{92}$$

with solution

$$U(\eta) = U_0 \cos\left(\sqrt{\gamma}\eta\right) + \frac{V_0 \sin\left(\sqrt{\gamma}\eta\right)}{\sqrt{\gamma}}, \tag{93}$$

$$V(\eta) = V_0 \cos\left(\sqrt{\gamma}\eta\right) - \sqrt{\gamma}U_0 \sin\left(\sqrt{\gamma}\eta\right). \tag{94}$$

Finally we have the parametric solution

$$a = \exp\left(\frac{U_0 \cos\left(\sqrt{\gamma}\eta\right) + \frac{V_0 \sin\left(\sqrt{\gamma}\eta\right)}{\sqrt{\gamma}} \cdots}{\sqrt{1 - \left(U_0 \cos\left(\sqrt{\gamma}\eta\right) + \frac{V_0 \sin\left(\sqrt{\gamma}\eta\right)}{\sqrt{\gamma}}\right)^2 - \left(V_0 \cos\left(\sqrt{\gamma}\eta\right) - \cdots\right.}}\right. \tag{95}$$

$$\dot{a} = \frac{B_0\left(V_0 \cos\left(\sqrt{\gamma}\eta\right) - \sqrt{\gamma}U_0 \sin\left(\sqrt{\gamma}\eta\right)\right) \cdots}{\sqrt{1 - \left(U_0 \cos\left(\sqrt{\gamma}\eta\right) + \frac{V_0 \sin\left(\sqrt{\gamma}\eta\right)}{\sqrt{\gamma}}\right)^2 - \left(V_0 \cos\left(\sqrt{\gamma}\eta\right) - \sqrt{\gamma}U \cdots\right.}} \tag{96}$$

which gives a periodic solution provided $\gamma > 0$. In the left panels of Figs. 9 and 10 we have not selected the value of

ρ_0 that makes (90) identically satisfied, but a closed value to show that the system admits a fixed point close to the origin that it is structurally stable.

On the other hand, in the same way as for (42)-(43), the asymptotic system near the fixed point at infinity for $U > 0, w_m > 0$ is

$$\frac{dU}{d\eta} = V^3, \quad \frac{dV}{d\eta} = -UV^2. \tag{97}$$

The orbits lying on the circle at infinity can be parametrized as

$$U(\eta) = \pm\frac{c_2 + \eta}{\sqrt{(c_2 + \eta)^2 + 1}}, \quad V(\eta) = \sqrt{\frac{1}{(c_2 + \eta)^2 + 1}}$$

or

$$U(\eta) = \pm\frac{\eta - c_2}{\sqrt{(\eta - c_2)^2 + 1}}, \quad V(\eta) = -\sqrt{\frac{1}{(\eta - c_2)^2 + 1}}.$$

Thus, there are fixed points lying on the circumference at infinity $(U, V) = (\pm 1, 0)$.

A. Integrability and connection with the observables

In this section we comment on the integrability of the system at hand, and calculate some observables in terms of redshift.

We assume that B is not a constant or trivial, and by introducing the logarithmic time variable $N = \ln a$, it is easy to recast the field equations as

$$\frac{dB}{dN} = -2B, \tag{98}$$

$$\frac{dH}{dN} = \frac{B^2 e^{-\frac{1}{2}\alpha B^2}\left(-4\beta + \alpha^2 B^4 + 2\alpha\beta B^2\right)}{3H\left(2\beta + \alpha B^2\right)^2} - \frac{1}{2}(w_m + 1)\frac{\rho_m}{H}, \tag{99}$$

$$\frac{d\rho_m}{dN} = -3(w_m + 1)\rho_m, \tag{100}$$

subject to the restriction

$$\frac{B^2 e^{-\frac{1}{2}\alpha B^2}}{2\beta + \alpha B^2} + \rho_m = 3H^2 \geq 0. \tag{101}$$

The above is integrable and gives

$$B(a) = \frac{c_1}{a^2}, \quad \rho_m(a) = c_2 a^{-3(w_m+1)}, \quad H(a) = \pm \frac{\sqrt{c_1^2 e^{-\frac{\alpha c_1^2}{2a^4}} + a^{-3(w_m}}}{} \tag{102}$$

In the cosmological applications with take the positive root for H.

From the compatibility condition

$$-\frac{B^2 e^{-\frac{1}{2}\alpha B^2}}{\alpha B^2 + 2\beta} + 3H^2 - \rho_m = 0, \tag{103}$$

it follows $c_3 = 0$. Hence, we have

$$H = \sqrt{\frac{1}{3}c_2 a^{-3(w_m+1)} + \frac{{c_1}^2 e^{-\frac{\alpha {c_1}^2}{2a^4}}}{3(2a^4\beta + \alpha {c_1}^2)}}. \tag{104}$$

The deceleration parameter is now given by

$$q = \frac{a c_2 (3w_m+1) e^{\frac{\alpha {c_1}^2}{2a^4}} \left(2a^4\beta + \alpha {c_1}^2\right)^2 - 2{c_1}^2 a^{3w_m} \left(-2a^8\beta + \alpha a^4 (2\ldots\right.}{2a\left(2a^4\beta + \alpha {c_1}^2\right)\left({c_1}^2 a^{3w_m+3} + c_2 e^{\frac{\alpha {c_1}^2}{2a^4}} \left(2a^4\beta + \alpha\ldots\right.\right.} \tag{105}$$

Taking the reference values $z = 0, a = 1$ for today. Defining $B(0) = B_0, H(0) = H_0, \Omega_{m,0} = \rho_m(0)/(3H_0^2)$, the current values of B, H, and the fractional energy density of matter the integration parameters are specified by

$$c_1 = B_0, c_2 = 3{H_0}^2\Omega_{m0}, \beta = -\frac{\alpha {B_0}^2}{2} + \frac{e^{-\frac{\alpha {B_0}^2}{2}}}{6{H_0}^2(1-\Omega_{m0})}. \tag{106}$$

In terms of redshift we have

$$B(z) = B_0(z+1)^2, \tag{107}$$

$$\rho_m(z) = 3{H_0}^2\Omega_{m0}(1+z)^{3(1+w_m)}, \tag{108}$$

$$H(z) = H_0\sqrt{\frac{{B_0}^2(1-\Omega_{m0})(z+1)^4 e^{-\frac{1}{2}\alpha {B_0}^2(z+1)^4}}{3\alpha {B_0}^2{H_0}^2(1-\Omega_{m0})z(z+2)(z(z+2)+2)+e^{-\frac{\alpha {B_0}^2}{2}}}} \tag{109}$$

and q is given by

$$q+1 = \frac{-2\alpha^2{B_0}^6(z+1)^9 - 4\alpha\beta{B_0}^4(z+1)^5 + 9(w_m+1)\Omega_{m0}(z+1)^{3wm}e^{\frac{1}{2}\alpha{B_0}^2(z+1)^4}\left(\alpha{B_0}^2H_0(z+1)^4+2\beta H_0\right)^2 + 8\beta{B_0}^2(z+1)}{2\left(2\beta+\alpha{B_0}^2(z+1)^4\right)\left(3H_0^2\Omega_{m0}(z+1)^{3w_m}e^{\frac{1}{2}\alpha{B_0}^2(z+1)^4}\left(2\beta+\alpha{B_0}^2(z+1)^4\right)+{B_0}^2(z+1)\right)}. \tag{110}$$

VI. OBSERVATIONAL CONSTRAINTS

In this section we put constraints on the NED parameters, i.e. α and B_0, using the observational Hubble data from cosmic chronometers and the latest SNIa data.

- Observational Hubble data (OHD). The differential age (DA) method measures $H(z)$ between two passively-evolving galaxies with similar metallicities and separated by a small redshift interval (cosmic chronometers) [89, 90]. The data provided by the DA method are cosmological-model-independent and then they can be used to probe alternative cosmological models. Here, we use the latest OHD obtained from DA, which contains 31 data points covering $0 < z < 1.97$, compiled by [91]. The figure-of-merit for the OHD is writ-

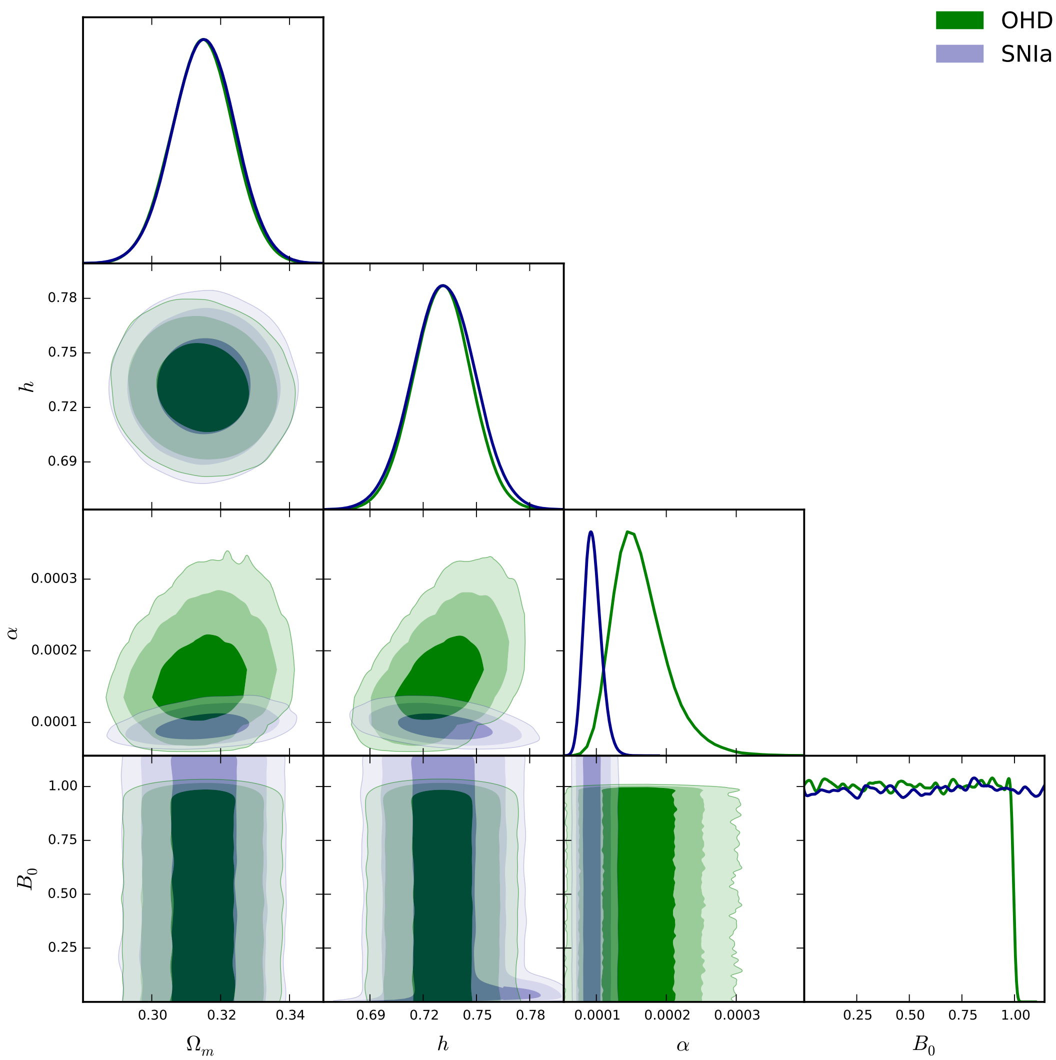

FIG. 11: 1D marginalized posterior distributions and the 2D 68%, 95%, 99.7% confidence levels for the Ω_{m0}, α, and B_0 parameters of the NED cosmology using OHD and SNIa data.

ten as

$$\chi^2_{\rm OHD} = \sum_{i=1}^{31} \frac{[H(z_i) - H_{da}(z_i)]^2}{\sigma^2_{H_i}} + \left(\frac{H_0 - 73.24}{1.74}\right)^2, \tag{111}$$

where $H(z_i)$ is the theoretical Hubble parameter (Eq. 109), $H_{da}(z_i)$ is the observational one at redshift z_i, and σ_{H_i} its error. Notice that in the last expression, we also consider the measurement of $H_0 = 73.24 \pm 1.74 \text{Kms}^{-1}\text{Mpc}^{-1}$

by [92] as a Gaussian prior.

- Type Ia Supernovae. The first evidence of the accelerating expansion of the Universe was provided by the observations of distant type Ia Supernovae (SNIa) [22, 23]. Over the last years, the high-resolution SNIa observations have demonstrated be a key cosmological probe due the shape their light curves can be standardizable. Thus, any alternative cosmological model should be confronted with the latest SNIa data. Here, we use the joint-light-analysis (JLA) compilation by [93] consisting in 740 data points in the range $0.01 < z < 1.2$. For the JLA sample, the observational distance modulus can be computed as

$$\mu_{obs} = m_B - (M_B - a\,X_1 + b\,C)\,, \tag{112}$$

where m_B is the observed peak magnitude in rest-frame B band, X_1 is the time stretching of the light-curve, C is the supernova color at maximum brightness, and M_B[2], a, and b are nuisance parameters in the distance estimate. On the other hand, the theoretical distance modulus is given by $\mu_{th} = 5\log_{10}(d_L/10pc)$, where d_L, the luminosity distance predicted by the NED cosmology (an interesting study of the SNIa luminosity distance in Born-Infeld NED cosmology is presented in [94]), reads as

$$d_L = (1+z)c\int_0^z \frac{\mathrm{dz}'}{H(z')}. \tag{114}$$

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[2] Notice that

$$M_b = \begin{cases} M_b^1, & \text{if the host stellar mass } M_{stellar} < 10^{10} M_\odot \\ M_b^1 + \delta_M, & \text{otherwise} \end{cases} \tag{113}$$

Therefore, the figure-of-merit for the SNIa data can be written as

$$\chi^2_{\text{SNIa}} = (\boldsymbol{\mu_{\text{obs}}} - \boldsymbol{\mu_{\text{th}}})^\dagger \text{Cov}(a,b)^{-1}(\boldsymbol{\mu_{\text{obs}}} - \boldsymbol{\mu_{\text{th}}}), \quad (115)$$

where $\text{Cov}(a,b)$ is the covariance matrix [3] of $\boldsymbol{\mu_{\text{obs}}}$ provided by [93].

We have tested two models for the NED cosmology: with and without including a perfect fluid. To constrain the free parameters, (Ω_{m0}, h, α, and B_0), we perform a Bayesian Monte Carlo Markov Chain (MCMC) analysis using the emcee python module using 800 walkers, 500 steps in the burn-in-phase, and 4000 MCMC steps to guaranty the convergence. We found that the first case is not able to fit the OHD. For the model including a fluid, we consider that it is dust matter with an equation of state $w_m = 0$ which associated with a dark matter component. Table I gives the mean values for the NED model parameters using different cosmological data. We obtain the narrow constraints $\alpha = 0.00015^{+4.24\times10^{-5}}_{-3.11\times10^{-5}}$ and $\alpha = 0.000093^{+1.29\times10^{-5}}_{-1.11\times10^{-5}}$ from OHD and SNIa data respectively. Nevertheless, the B_0 parameter is weakly constrained by both data. Figure 11 shows the 1D marginalized posterior distribution and the 2D confidence contours for the Ω_m, h, α, and B_0 parameters for the NED cosmology. Notice that both OHD and SNIa data provide consistent constraints on the NED parameters. Figure 12 illustrates the fitting of the NED cosmology to OHD using the parameter mean values given in Table I.

[3] available at http://supernovae.in2p3.fr/sdss_snls_jla/ReadMe.html

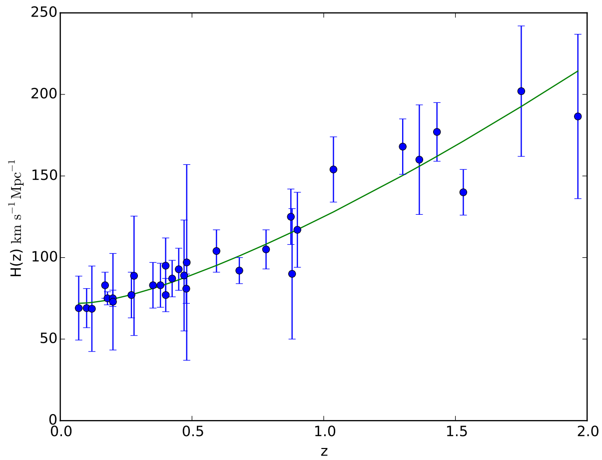


FIG. 12: Fitting to OHD data (dots) using the Ω_{m0}, h, α, and B_0 mean values into the theoretical $H(z)$ (solid line) given by Eq. 109.

NED model

Data set	χ^2_{min}	Ω_{m0}	h	$\alpha(10^{-5})$	B_0
OHD	14.90	$0.31^{+0.008}_{-0.008}$	$0.73^{+0.01}_{-0.01}$	$15^{+4.24}_{-3.11}$	$0.59^{+0.40}_{-0.40}$
SNIa	683.20	$0.31^{+0.008}_{-0.008}$	$0.73^{+0.01}_{-0.01}$	$9.39^{+1.29}_{-1.11}$	$0.60^{+0.40}_{-0.41}$

TABLE I: Mean values for the NED parameters (Ω_{m0}, h, α, and B_0) derived from OHD and SNIa data (JLA sample). We estimate $M_B^1 = -18.95^{+0.05}_{-0.05}$, $\delta_M = -0.07^{+0.02}_{-0.02}$, $a = 0.14^{+0.006}_{-0.06}$, and $b = 3.10^{+0.08}_{-0.07}$ when the SNIa data are used.

The reconstruction of the deceleration parameter $q(z)$ in the range $0 < z < 2$ for OHD (top panel) and SNIa data (bottom panel) is also shown in Figure 13. Notice that $q(z) \to 1/2$ when $z \to \infty$. In addition, both cosmological data predict an accelerating expansion, i.e., the Universe passes of a decelerated phase to an accelerated stage at redshift ~ 0.5 and ~ 0.7 for the OHD and SNIa constraints respectively. Therefore, the NED cosmology including dust matter ($w_m = 0$) is able to drive the late-time cosmic acceleration of the standard cosmological model.

A. The interplay between dynamical systems approach and Bayesian Monte Carlo Markov Chain analysis

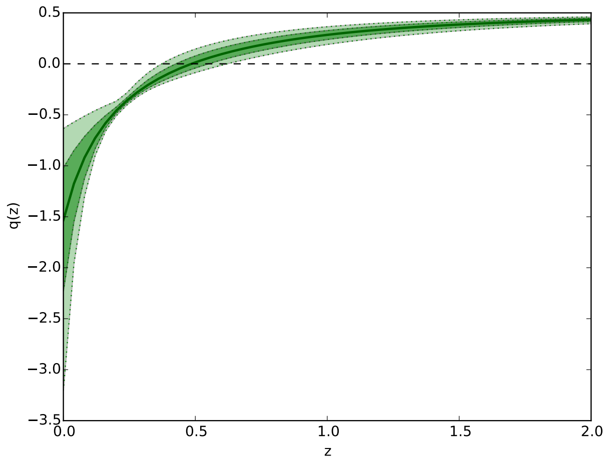
Combining the above results on the parameter α: $\alpha = \begin{cases} \left(15^{+4.24}_{-3.11}\right) \times 10^{-5} & \text{OHD}, \\ \left(9.39^{+1.29}_{-1.11}\right) \times 10^{-5} & \text{SNIa} \end{cases}$, we can find from the equations of motion the approximated equations

$$\dot{B} = -2BH, \dot{H} = -\frac{2B^2 + 3\beta\rho_m}{6\beta}, \dot{\rho}_m = -3H\rho_m,$$
$$-\frac{B^2}{2\beta} + 3H^2 - \rho_m = 0, \tag{116}$$

by taking the limit $\alpha = 0$, and assuming dust matter. This system is consistent with a model Universe with just radiation $\rho_R = \frac{B^2}{2\beta} = \frac{B_0}{\beta a^4}$, with density today $\rho_{R,0} = \frac{B_0}{\beta}$, and dust matter $\rho_m = \frac{\rho_{m,0}}{a^3}$, as expected for a model for the early time dynamics. Observe that from $\beta = \frac{1}{6{H_0}^2(1-\Omega_{m0})}$ it follows $\rho_{R,0} = \frac{B_0}{\beta} = 6B_0{H_0}^2(1-\Omega_{m0}) = 2B_0\rho_{R,0}$, and this is consistent with the mean value $B = 0.5$. The result is that the are not statistical differences with the usual model during the radiation epoch which holds for $\alpha = 0$. However, taking α slightly different from zero, we find that the NED with dust matter ($w_m = 0$) is able to drive the late-time cosmic acceleration of the standard cosmological model as shown in Fig. 13.

VII. CONCLUSION

In this paper, we have considered a new field of NED as a source of gravity to shed light on the dynamics behind the



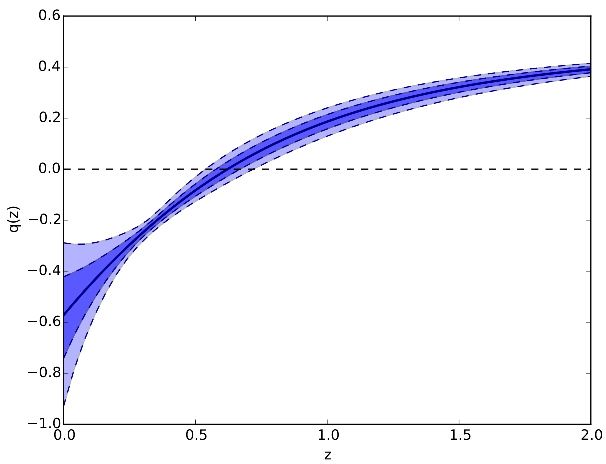


FIG. 13: In the top (bottom) panel we show the reconstruction of the deceleration parameter $q(z)$ using using the Ω_{m0}, h, α, and B_0 mean values from OHD (SNIa) into the theoretical $q(z)$ given by (110). The shadowed areas represent the 1σ and 2σ confidence levels.

accelerating Universe and solve the singularity problem of the Big Bang and the Universe curvature.

We have argued that the NED on cosmological scales could reveal the acceleration of the Universe during the inflationary era. We have also shown that, after this period of cosmic inflation the Universe undergoes decelerated expansion and asymptotically approaches the Minkowski spacetime.

From the dynamical system approach we have found for parameters $\alpha = 2B_0^{-2}, \beta = -2$, and assuming no background matter, the realization of a cyclic Universe in our model supported by NED near the value $a_0 = 1$. The scale factor goes below and above the value $a = 1$, reaching a maximum and a minimum value of a, and a is bounded away zero (there is no initial singularity, as expected from our NED proposal). For our two models (not including matter, and for prefect fluid), We have found approximated solutions in parametric form for the scalar factor and its time derivative, which are valid in the neighborhood of the fixed point at the finite region. For some values of the parameters we found also the realization of a cyclic Universe.

For the model without matter, we have found solutions with B constant ($\alpha B^2 = -\beta + \sqrt{\beta^2 + 4\beta}$), which supports the inflation similar to de-Sitter Universe, with

$$a \propto e^{\sqrt{\Lambda} t}, \quad \Lambda = \frac{e^{\frac{1}{2}\left(\beta - \sqrt{\beta(\beta+4)}\right)} \left(\beta - \sqrt{\beta(\beta+4)} + 2\right)}{6\alpha}, \tag{117}$$

provided $\beta > 0, \alpha > 0$.

In the case of evolving H, it was provided explicit expressions for $H(z)$ by direct integration of the equations of motion in the two models. Using these expressions for $H(z)$ we have tested our two models for the NED cosmology using OHD and SNIa data: with and without including a perfect fluid. To constrain the parameters we have performed a Bayesian Monte Carlo Markov Chain (MCMC) analysis using the emcee python module using 800 walkers, 500 steps in the burn-in-phase, and 4000 MCMC steps. We have found that the first case is not able to fit the OHD. For the model including a fluid, we consider that it is dust matter with an equation of state $w_m = 0$ which associated with a dark matter component. In this particular example we conclude that the model is a good model for the early time Universe, but there are not statistical differences with the usual model for the radiation epoch. We have reconstructed the deceleration parameter $q(z)$ in the range $0 < z < 2$ as shown in Figure 13 for OHD (top panel) and SNIa data (bottom panel). Notice that $q(z) \to 1/2$ when $z \to \infty$. In addition, both cosmological data predict an accelerating expansion, i.e., the Universe passes of a decelerated phase to an accelerated stage at redshift ~ 0.5 and ~ 0.7 for the OHD and SNIa constraints respectively. That is, first we have proved that a model based on NED alone does not passes the observational tests, in particular Observational Hubble data and type Ia Supernovae data. Second, we have proved that dust matter plus NED passed these tests, and provide late-time acceleration (the combined effect of both matter contributions).

Summarizing, using in a combined way the powerful of phase-space analysis, and the observational fit, one is able to falsify (in the sense of test) a theoretical cosmological model based on NED. From one side, dynamical systems tools allows to identify regions in the parameter space to provide stability conditions for fixed points with physical meaning. Furthermore, it was possible to use 2D phase spaces, showing trajectories in a geometrical way, so that it becomes easy to observe the property with the help of the attractors which are the most easily seen experimentally. On the other hand, comparison with data allows to refine more the region of parameters.

ACKNOWLEDGEMENTS

This work was supported by the Chilean FONDECYT Grant No. 3170035 (AÖ) and FONDECYT Grant No. 3160674 (JM). GL thanks to Department of Mathematics at Universidad Catolica del Norte for warm hospitality and financial support. AÖ is grateful to the CERN theory (CERN-TH) division for hospitality where part of this work was done.

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[1] S. M. Carroll, Living Rev.Rel. **4**, 1 (2001).
[2] M. Lopez-Corredoira, Found.Phys. **47** no.6, 711-768 (2017).
[3] V. F. Mukhanov and R. H. Brandenberger, Phys. Rev. Lett. **68**, 1969 (1992).
[4] S. I. Kruglov, Phys.Rev. D **92**, no.12, 123523 (2015).
[5] S. I. Kruglov, Int. J. Mod. Phys. D **25**, no. 4, 1640002 (2016).
[6] S. I. Kruglov, Int. J. Mod. Phys. A **32**, 13, 1750071 (2017).
[7] A. Övgün, Eur. Phys. J. C **77**, no. 2, 105 (2017)
[8] M. Sharif and S. Mumtaz, Eur.Phys.J. C **77**, no.2, 136 (2017).
[9] D. Huterer and D. L Shafer, arXiv:1709.01091.
[10] R. Durrer, A. Neronov, Astron Astrophys Rev **21**, 62, (2013).
[11] S. H. Hendi, JHEP **1203**, 065 (2012).
[12] V. A. De Lorenci, R. Klippert, M. Novello and J. M. Salim, Phys. Rev. D **65**, 063501 (2002).
[13] M. Novello, et al. Phys.Rev. D **69**, 127301 (2004).
[14] M. Novello, et al. Class.Quant.Grav. **24**, 3021-3036 (2007).
[15] D. N.Vollick, Phys.Rev. D **78,** 063524 (2008).
[16] M. Novello, S.E.P. Bergliaffa, Phys.Rept. **463**, 127-213 (2008).
[17] V.F. Antunes, M. Novello, Grav.Cosmol. **22**, no.1, 1-9 (2016).
[18] E. Bittencourt, U. Moschella, M. Novello, J.D. Toniato, Phys.Rev. D **90**, no.12, 123540 (2014).
[19] K. E. Kunze, Plasma Phys.Control.Fusion **55**, 124026 (2013).
[20] K. E. Kunze, Phys.Rev. D **77**, 023530 (2008).
[21] L. Campanelli, P. Cea, G. L. Fogli and L. Tedesco, Phys. Rev. D **77**, 043001 (2008).
[22] A. G. Riess et al., A.J. **116** 1009-1038 (1988).
[23] S. Perlmutter et al. (Supernova Cosmology Project), Astrophys. J. **517** 565-586 (1999).
[24] K. Dutta, S. Panda and A. Patel, Phys.Rev. D **94** no.2, 024016 (2016).
[25] R. Garcia-Salcedo and N. Breton, Class. Quant. Grav. **20**, 5425 (2003)
[26] E. N. Saridakis, M. Tsoukalas, arXiv:1601.06734.
[27] H. Sheikhahmadi, E.N. Saridakis, A. Aghamohammadi, K. Saaidi, arXiv:1603.03883.
[28] R. Garcia-Salcedo, et al. Int.J.Mod.Phys. A**15**, 4341-4354 (2000).
[29] C.S. Camara, et al. Phys.Rev. D **69**, 123504 (2004).

[30] E. Elizalde, et al. Phys.Lett. B **574**, 1-7 (2003).
[31] C. Quercellini , M. Bruni, A. Balbi, D. Pietrobon, Phys.Rev. D **78**, 063527 (2008).
[32] H. Azri and D. Demir, Phys.Rev. D **95** no.12, 124007 (2017).
[33] A.W. Beckwith, J.Phys.Conf.Ser. **626,** no.1, 012058 (2015).
[34] Y.Cai , E.N. Saridakis, M. R. Setare, J. Xia, Phys.Rept. **493**, 1-60 (2010).
[35] E.N. Saridakis, S. V. Sushkov, Phys.Rev. D **81**, 083510 (2010).
[36] E.N. Saridakis, Eur.Phys.J. C **67**, 229-235 (2010).
[37] N. Breton, R. Lazkoz, A. Montiel, JCAP **1210**, 013 (2012).
[38] G. Leon and E. N. Saridakis, JCAP **0911**, 006 (2009).
[39] C. Xu, E. N. Saridakis and G. Leon, JCAP **1207**, 005 (2012).
[40] G. Leon and E. N. Saridakis, JCAP **1303**, 025 (2013).
[41] G. Leon, J. Saavedra and E. N. Saridakis, Class. Quant. Grav. **30**, 135001 (2013).
[42] G. Kofinas, G. Leon and E. N. Saridakis, Class. Quant. Grav. **31**, 175011 (2014).
[43] G. Leon, Y. Leyva and J. Socorro, Phys. Lett. B **732**, 285 (2014).
[44] C. R. Fadragas, G. Leon and E. N. Saridakis, Class. Quant. Grav. **31**, 075018 (2014).
[45] G. Leon and E. N. Saridakis, JCAP **1511**, no. 11, 009 (2015).
[46] G. Pulgar, J. Saavedra, G. Leon and Y. Leyva, JCAP **1505**, 046 (2015).
[47] G. Leon and E. N. Saridakis, JCAP **1504**, no. 04, 031 (2015).
[48] A. Giacomini, S. Jamal, G. Leon, A. Paliathanasis and J. Saavedra, Phys. Rev. D **95**, no. 12, 124060 (2017).
[49] A. Joyce, L. Lombriser and F. Schmidt, Ann. Rev. Nucl. Part. Sci. **66**, 95 (2016).
[50] S. Tsujikawa, Class. Quant. Grav. **30**, 214003 (2013).
[51] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rept. **692**, 1 (2017).
[52] S. Lepe, G. Otalora and J. Saavedra, Phys. Rev. D **96**, no. 2, 023536 (2017).
[53] G. Otalora, Int. J. Mod. Phys. D **25**, no. 02, 1650025 (2015).
[54] T. Harko, F. S. N. Lobo, G. Otalora and E. N. Saridakis, JCAP **1412**, 021 (2014).
[55] G. Otalora, Phys. Rev. D **88**, 063505 (2013).
[56] G. Otalora, JCAP **1307**, 044 (2013).
[57] L. Karpathopoulos, S. Basilakos, G. Leon, A. Paliathanasis and M. Tsamparlis, arXiv:1709.02197 [gr-qc].
[58] O. Akarsu, M. Arik and N. Katirci, Found. Phys. **47**, no. 6, 769 (2017).
[59] O. Akarsu, M. Arik, N. Katirci and M. Kavuk, JCAP **1407**, 009 (2014).
[60] T. Tajima, S. Cable, K. Shibata, and R. M. Kulsrud, Astrophys. J. **390**, 309 (1992).
[61] M. Giovannini and M. Shaposhnikov, Phys. Rev. D **57**, 2186 (1998).
[62] A. Campos and B. L. Hu, Phys. Rev. D **58**, 125021 (1998).
[63] G.G. Dunne and T. Hall, Phys. Rev. D **58**, 105022 (1998).
[64] M. Joyce and M. Shaposhnikov, Phys. Rev. Lett. **79**, 1193 (1997) .
[65] M. Novello, S. E. Perez Bergliaffa and J. M. Salim, Class. Quant. Grav. **17**, 3821 (2000)
[66] Z. Haba, A. Stachowski and M. Szydlowski, JCAP **1607**, no. 07, 024 (2016).
[67] A. Stachowski and M. Szydlowski, Eur. Phys. J. C **76**, no. 11, 606 (2016).
[68] O. Hrycyna, M. Szydlowski and M. Kamionka, Phys. Rev. D **90**, no. 12, 124040 (2014).
[69] M. Szydlowski, Astrophys. Space Sci. **339**, 389 (2012).
[70] M. Szydlowski and O. Hrycyna, JCAP **0901**, 039 (2009).
[71] M. P. Dabrowski, T. Stachowiak and M. Szydlowski, Phys. Rev. D **68**, 103519 (2003).
[72] M. Szydlowski, A. Krawiec, A. Kurek and M. Kamionka, Eur. Phys. J. C **75**, no. 99, 5 (2015).
[73] M. Szydlowski, A. Kurek and A. Krawiec, Phys. Lett. B **642**, 171 (2006).
[74] W. Godlowski and M. Szydlowski, Phys. Lett. B **623**, 10 (2005).
[75] M. Szydlowski, Phys. Lett. B **632**, 1 (2006).
[76] L. Karpathopoulos, S. Basilakos, G. Leon, A. Paliathanasis and M. Tsamparlis, arXiv:1709.02197.
[77] M.Szydlowski and P. Tambor, arXiv:0805.2665.
[78] V. A. De Lorenci, R. Klippert, M. Novello and J. M. Salim, Phys. Lett. B **482**, 134 (2000).
[79] M. Novello, V. A. De Lorenci, J. M. Salim and R. Klippert, Phys. Rev. D **61**, 045001 (2000).
[80] M. Novello and J. M. Salim, Phys. Rev. D **20**, 377 (1979).
[81] G. A. Mourou, T. Tajima and S. V. Bulanov, Rev. Mod. Phys. **78**, 309 (2006).
[82] V. A. De Lorenci and R. Klippert, Phys. Rev. D **65**, 064027 (2002).
[83] S. Villalba-Chavez and A. E. Shabad, Phys. Rev. D **86**, 105040 (2012).
[84] F. Karbstein, H. Gies, M. Reuter and M. Zepf, Phys. Rev. D **92**, no. 7, 071301 (2015).
[85] R. Tolman, P. Ehrenfest, Phys.Rev. **36**, no.12, 1791 (1930).
[86] A. Balbi, EPJ Web Conf. **58**, 02004 (2013).
[87] R. Garcia-Salcedo, T. Gonzalez, I. Quiros, Phys.Rev. D **89**, no.8, 084047 (2014).
[88] M. Kierdorf et al., Astronomy & Astrophysics **600**, id.A18 (2017).
[89] R. Jimenez and A. Loeb, Astrophys. J. **573**, 37 (2002) doi:10.1086/340549 [astro-ph/0106145].
[90] Moresco, M., Cimatti, A., Jimenez, R., et al. 2012, JCAP, 8, 006.
[91] J. Magana, M. H. Amante, M. A. Garcia-Aspeitia and V. Motta, arXiv:1706.09848 [astro-ph.CO].
[92] A. G. Riess *et al.*, Astrophys. J. **826**, no. 1, 56 (2016) doi:10.3847/0004-637X/826/1/56 [arXiv:1604.01424 [astro-ph.CO]].
[93] Betoule, M., Kessler, R., Guy, J., et al. 2014, Astron. & Astroph., 568, A22

[94] Aiello, M., Bengochea, G. R., & Ferraro, R. 2008, JCAP, 6, 006