String-Inspired Gravity through Symmetries

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Idea de la charla

- Modelos de super cuerdas MODIFICADAS
- Obtener soluciones exactas mediante simetrías

  - Autosimilaridad, Colineaciones de materia
  - Grupos de Lie
  - Simetrías tipo Noether, colineadores del Lagrangiano

- Ejemplos. Soluciones SS

  - FRW
  - BII

Modelos de super-cuerdas modificadas

Action in string frame

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}e^{-\phi}\left[R+(\nabla\phi)^2-V-\frac{1}{12}\mathbf{H}^2\right]+\int d^4x\sqrt{-g}\mathscr{L}_{\text{matter}},$$

- $T-$duality property $\left(a(t)\to a^{-1}(-t)\right)$ pre-BB
- Si $V=V_0$ NO existen soluciones SS

Ecuaciones de Campo

$$G_{\mu}^{\nu}={}^{\text{eff}}\,T_{\mu}^{\nu},\qquad {}^{\text{eff}}T_{\mu}^{\nu}={}^{(m)}\,T_{\mu}^{\nu}+{}^{(\phi)}\,T_{\mu}^{\nu}+{}^{(\mathbf{H})}\,T_{\mu}^{\nu}+{}^{(V)}\,T_{\mu}^{\nu},$$

$$R_{\mu}^{\nu}-\frac{1}{2}g_{\mu}^{\nu}R=e^{\phi}T_{\mu}^{\nu}+\frac{1}{12}\left(3\mathbf{H}_{\mu\lambda\kappa}\mathbf{H}^{\nu\lambda\kappa}-\frac{1}{2}g_{\mu}^{\nu}\mathbf{H}^2\right)-\frac{1}{2}g_{\mu}^{\nu}V$$
$$-\frac{1}{2}g_{\mu}^{\nu}\left(\nabla\phi\right)^2+\left(g_{\mu}^{\nu}g^{\lambda\kappa}-g_{\mu}^{\lambda}g^{\nu\kappa}\right)\nabla_{\lambda}\nabla_{\kappa}\phi\ ,$$

Modelos de super-cuerdas modificadas

$\mathbf{H}_{\alpha\beta\delta}$, the Kalb-Ramond field, is the completely antisymmetric tensor field strength defined by $\mathbf{H} = dB$, where B is a rank-two antisymmetric tensor. Variations in $B_{\mu\nu}$

$$\nabla_{\mu}\left(e^{-\phi}\mathbf{H}^{\mu\nu\lambda}\right) = 0,$$

assume

$$\mathbf{H}^{\mu\nu\lambda} = e^{\phi}\varepsilon^{\mu\nu\lambda\kappa}h_{,\kappa}, \quad \Box h + \nabla^{\mu}\phi\nabla_{\mu}h = 0,$$

Variation in ϕ

$$2\Box\phi + R - (\nabla\phi)^2 - V + \partial_{\phi}V - \frac{1}{12}\mathbf{H}^2 = 0,$$

assume

$$\nabla_{\nu}\left({}^{(m)}T_{\mu}^{\nu}\right) = 0,$$

Modelos de super-cuerdas modificadas

$$^{\text{eff}}T_{\mu}^{\nu} = ^{(m)}T_{\mu}^{\nu} + ^{(\phi)}T_{\mu}^{\nu} + ^{(\mathbf{H})}T_{\mu}^{\nu} + ^{(V)}T_{\mu}^{\nu},$$

$$^{(m)}T_{\mu}^{\nu} = \kappa^2 e^{\phi} T_{\mu}^{\nu},$$

$$^{(\phi)}T_{\mu}^{\nu} = -\frac{1}{2} g_{\mu}^{\nu} (\nabla\phi)^2 + \left(g_{\mu}^{\nu} g^{\lambda\kappa} - g_{\mu}^{\lambda} g^{\nu\kappa} \right) \nabla_{\lambda} \nabla_{\kappa} \phi,$$

$$^{(\mathbf{H})}T_{\mu}^{\nu} = \frac{1}{12} \left(3\mathbf{H}_{\mu\lambda\kappa} \mathbf{H}^{\nu\lambda\kappa} - \frac{1}{2} g_{\mu}^{\nu} \mathbf{H}^2 \right),$$

$$^{(V)}T_{\mu}^{\nu} = -\frac{1}{2} g_{\mu}^{\nu} V,$$

We prove that the matter conservation condition is verified for the FE

Theorem

FE verify the condition, $\nabla_{\nu} T_{\mu}^{\nu} = 0$, that is, there is matter conservation.

Métodos matemáticos

- Simetrías (Autosimilaridad)

  - Geométrica, Autosimilaridad
  - Colineaciones de materia

- Grupos de Lie (Ecuación de Klein-Gordon)
- Simetrías tipo Noether

  - Colineaciones del Lagrangiago: $L_X \mathscr{L} = 0$

Autosimilaridad

La autosimilaridad es definida por la existencia de un campo homotético, $\mathscr{H} \in \mathfrak{X}(M)$, en el espacio-tiempo que verifica la ecuación

$$L_{\mathscr{H}} g_{ij} = 2\alpha g_{ij}, \quad \alpha = const.$$

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- La existencia de $\mathscr{H} \in \mathfrak{X}(M)$, implica que los factores de escala de la métrica siguen una ley tipo potencias i.e. $a(t) = a_0 t^{a_1}$ con $(a_i)_{i=1}^3 \in \mathbb{R}^+$,
  Restricciones geométricas

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  Restricciones geométricas
- Punto de vista dinámico:
  - Soluciones autosimilares corresponden a puntos críticos (SD)
  - Describen el comportamiento asintótico de soluciones más complejas (papel fundamental en los modelos Bianchi)

Colineaciones de materia

Decimos que un campo vectorial $V \in \mathfrak{X}(M)$, (con flujo F_t) es una colineación de materia (CM), si la derivada de Lie del tensor efectivo de energía-momento T_{ij}^{eff}, a lo largo de dicho campo es nula, es decir,

$$L_V T_{ij}^{eff} = 0, \qquad F_t(T_{ij}^{eff}) = T_{ij}^{eff}.$$

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- No estamos interesados en estudiar el álgebra V
- Si ξ^a es un campo de Killing (o un campo homotético), entonces se verifica $L_\xi T_{ij} = 0$; por tanto, toda isometría es una CM, pero el inverso es falso en general.
- $L_{\mathscr{H}} T_{ij} = 0$, siendo $\mathscr{H}$ es un campo homotético y T_{ij} un tensor energía-momento efectivo, buscar las relaciones funcionales entre las distintas magnitudes físicas, ecuaciones de estado, de tal forma que las ecuaciones de campo admitan soluciones autosimilares

Grupos de Lie

$$y^{(n)} = \omega(x, y, y', \ldots.., y^{(n-1)}), \qquad y^{(k)} = \frac{d^k y}{dx^k},$$

Asumimos que la ODE admite un grupo uni-paramétrico G

$$\hat{x} = x + \varepsilon\xi + O(\varepsilon^2), \qquad \hat{y} = y + \varepsilon\eta + O(\varepsilon^2),$$

cuyo generador infinitesimal es: $X = \xi\,(x,y)\,\partial_x + \eta\,(x,y)\,\partial_y$.

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Teorema

Sea $X = \xi\,(x,y)\,\partial_x + \eta\,(x,y)\,\partial_y$ el generador infinitesimal del grupo G y sea

$$X^{(n)} = \xi\,(x,y)\,\partial_x + \eta\,(x,y)\,\partial_y + \eta^{(1)}\,(x,y,y_1)\,\partial_{y_1} + \ldots + \eta^{(n)}\,(x,y,y_1,..,y_n)\,\partial_{y_n},$$

su prolongación de orden n. Entonces, el grupo G deja invariante la ODE sii

$$X^{(n)}\left(y^{(n)} - \omega(x,y,y',..,y^{(n-1)})\right) = 0,$$

Grupos de Lie. Ejemplo

Ecuación de Endem modificada admite 8 simetrías

$$\ddot{x}+3x\dot{x}+x^3=0,$$

$$V_1=\frac{\partial}{\partial t},\ldots V_8=t^3\left(xt-2\right)\frac{\partial}{\partial t}+t\left(4-x\left(6t-4t^2x+t^3x^2\right)\right)\frac{\partial}{\partial x},$$

Estudiamos el álgebra y vemos que los campos interesantes son

$$V_2=x\frac{\partial}{\partial t}-x^3\frac{\partial}{\partial x},\qquad V_5=xt\frac{\partial}{\partial t}+x^2\left(1-xt\right)\frac{\partial}{\partial x}.$$

Buscamos un c.v. $(t,x)\longrightarrow(r,s)$ tal que: $Vr=0$, $Vs=1$. Por ejemplo:

1. V_2 induce el cv: $r=t-\frac{1}{x}$, $s=-\frac{1}{2x^2}$, entonces: $\ddot{s}=-1$.
2. V_5 induce el cv: $r=\frac{t}{x}-\frac{t^2}{2}$, $s=t-\frac{1}{x}$, así: $\ddot{s}=0$.

Ecuación de Endem re-modificada

$$\ddot{x}+f(x)\dot{x}+g(x)=0,$$

posibles formas de $f(x)$ y $g(x)$ para que la ODE sea integrable. Imponer una simetría y determinar $f(x)$ y $g(x)$.

Grupos de Lie

El conocimiento de una única simetría $X=\xi\partial_x+\eta\partial_y$ de (ODE) nos permite obtener una solución particular como un invariante del operador X, es decir, una solución de

$$\frac{dx}{\xi(x,y)}=\frac{dy}{\eta(x,y)} \qquad \Longrightarrow \qquad y'=\frac{\eta}{\xi}.$$

Esta solución particular se conoce con el nombre de solución invariante (generalización de las soluciones autosimilares obtenidas mediante análisis dimensional).

Definición

Decimos que $y=\phi(x)$ es una solución invariante de (ODE) si $y=\phi(x)$ es una curva invariante de X i.e. $X(y-\phi)=0$ y además es solución, particular, de (ODE).

Colineaciones de Materia vs Grupos de Lie

- La existencia de $\mathscr{H} \in \mathfrak{X}(M)$, implica que los factores de escala de la métrica siguen una ley tipo potencias i.e. $a(t) = a_0 t^{a_1}$, obtendremos restricciones sobre a_i
- CM restricciones sobre magnitudes físicas compatibles con soluciones autosimilares
- GL soluciones tipo potencias (power-law) más generales que las autosimilares
- GL más genérico, puede ser útil para el estudio de modelos mucho más complejos

Simetrías tipo Noether. Colineadores del Lagrangiano

Sea $\mathscr{L} = \mathscr{L}(q_n, \dot{q}_n)$, verificando las ecuaciones de EL: $\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) = \frac{\partial \mathscr{L}}{\partial q}$,

$$\frac{d}{dt}\left(X^n \frac{\partial \mathscr{L}}{\partial \dot{q}}\right) = X^n \frac{\partial \mathscr{L}}{\partial q} + \frac{dX^n}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} = L_X \mathscr{L}, \qquad X = X^n \frac{\partial}{\partial q} + \dot{X}^n \frac{\partial}{\partial \dot{q}},$$

siendo X es un VF ("estirado").
$X^n \frac{\partial \mathscr{L}}{\partial \dot{q}}$ se reescribe mediante la 1-forma Cartan $\theta_{\mathscr{L}} = \frac{\partial \mathscr{L}}{\partial \dot{q}} dq_n$, entonces

$$i_X \theta_{\mathscr{L}} = X^n \frac{\partial \mathscr{L}}{\partial \dot{q}} \qquad \Longrightarrow \qquad \frac{d}{dt}\left(i_X \theta_{\mathscr{L}}\right) = L_X \mathscr{L}.$$

$$i_X \theta_{\mathscr{L}} = \Sigma_0, \qquad \text{tal que} \qquad \frac{d\Sigma_0}{dt} = 0.$$

Teorema

Si X es una simetría de $\mathscr{L}$ entonces $i_X \theta_{\mathscr{L}} = \Sigma_0 = const$.

X, induce nuevas variables: $q_n = q_n(Q_k,)$, $\quad i_X Q_1 = 1$, $\quad i_X Q_2 = 0$,

$$\frac{\partial}{\partial Q_1}\left(\bar{\mathscr{L}}\left(q_n\left(Q_n\right), \dot{q}_n\left(Q_n, \dot{Q}_n\right)\right)\right) = 0,$$

$\bar{\mathscr{L}}$ ($\mathscr{L}$ en las nuevas variables) es cíclica en Q_1, solución exacta.

Simetrías tipo Noether

- Colineaciones Lagrangiano, cambios de variable que permiten obtener soluciones generales
- GL, más simetrías, pero sólo soluciones invariantes
- Relación con soluciones autosimilares
- Restricciones, válido para una sola geometría
- Las simetrías tipo Noether forman un subconjunto de las de Lie
- Sistemas EDP Noether muy complicados. Complicado encontrar soluciones

Soluciones Autosimilares

We consider a FRW metric, thus the HVF yields

$$\mathcal{H} = t\partial_t + (1-a_1)\left(x\partial_x + y\partial_y + z\partial_z\right),$$

where $a_1 \in \mathbb{R}$, is a numerical constant, The HVF for the BII metric yields

$$\mathcal{H} = t\partial_t + (1-a_1)\, x\partial_x + (1-a_2)\, y\partial_y + (1-a_3)\, z\partial_z,$$

with $a_1, a_2, a_3 \in \mathbb{R}$, where $a_2 + a_3 = 1 + a_1$.

Theorem

The FE admit SS solutions if the physical quantities take the following form:

$$\rho = \rho_0 t^{-(2+\phi_0)}, \qquad \phi = \phi_0 \ln t, \qquad h = h_0 t^{-\phi_0} + h_1, \qquad V = \Lambda e^{-\frac{2}{\phi_0}\phi},$$

where $\rho_0, \phi_0, h_0, h_1, \Lambda$ are constants.

Idea de la demostración. We split the effective stress-energy tensor in the following components:

$$\begin{aligned}
{}^{(m)}T_{\mu}^{\nu} &= e^{\phi}T_{\mu}^{\nu},\\
{}^{(\phi)}T_{\mu}^{\nu} &= \frac{1}{2}g_{\mu}^{\nu}\left[2\nabla^{2}\phi-(\nabla\phi)^{2}\right]-\nabla_{\mu}\nabla^{\nu}\phi,\\
{}^{(\mathbf{H})}T_{\mu}^{\nu} &= \frac{1}{12}\left(3\mathbf{H}_{\mu\lambda\kappa}\mathbf{H}^{\nu\lambda\kappa}-\frac{1}{2}g_{\mu}^{\nu}\mathbf{H}^{2}\right),\\
{}^{(V)}T_{\mu}^{\nu} &= -\frac{1}{2}g_{\mu}^{\nu}V.
\end{aligned}$$

In the string frame we calculate the following equations ($L_{\mathscr{H}}^{(i)}T_{\mu\nu}=0$);

$$\begin{aligned}
t\rho\phi'+t\rho'+2\rho &= 0, && L_{\mathscr{H}}^{(m)}T_{\mu\nu}=0,\\
t\left(\phi''\left(3H-\phi'\right)+3\phi'\left(\frac{a''}{a}-H^{2}\right)\right)+6\phi'H-\phi'^{2} &= 0, && L_{\mathscr{H}}^{(\phi)}T_{\mu\nu}=0,\\
th'\phi'+th''+h' &= 0, && L_{\mathscr{H}}^{(\mathbf{H})}T_{\mu\nu}=0,\\
V_{\phi}t\phi'+2V &= 0, && L_{\mathscr{H}}^{(V)}T_{\mu\nu}=0.
\end{aligned}$$

Grupos de Lie. Primera solución

We study the equation

$$2\Box\phi + R - (\nabla\phi)^2 - V + \partial_\phi V - \frac{1}{12}\mathbf{H}^2 = 0,$$

that we rewrite as follows

$$2\phi'' + 2K\phi'\theta - R - \phi'^2 + \frac{1}{2}e^{2\phi}h'^2 + V - V_\phi = 0,$$

where $\theta = u^i_{;i}$, $u^i = (1,0,0,0)$. We use the notation $\phi' = \frac{d}{dt}\phi$, $V_\phi = \frac{d}{d\phi}V$, etc.

Theorem

FE admit power-law solutions if the physical quantities take the following form:

$$\phi = \phi_0 \ln t, \qquad h = h_0 t^{-\phi_0} + h_1, \qquad V = \Lambda e^{-\frac{2}{\phi_0}\phi}.$$

Grupos de Lie. Primera solución

$$2\phi'' + 2K\phi'\theta - R - \phi'^2 + \frac{1}{2}e^{2\phi}h'^2 + V - V_\phi = 0,$$

Proof.

By studying through the LG method we get

$$\begin{gathered}
2\xi_{\phi\phi} + \xi_\phi = 0, \\
8K\theta\xi_\phi - 2\eta_\phi + 4\eta_{\phi\phi} - 8\xi_{t\phi} = 0, \\
3\left(2V - 2R - 2V_\phi + e^{2\phi}h'^2\right)\xi_\phi + 4K\left(\theta\xi_t + \xi\theta'\right) - 4\eta_t + 8\eta_{t\phi} - 4\xi_{tt} = 0, \\
2e^{2\phi}\left(h'h''\xi + h'^2\left(\xi_t - \eta_\phi + \eta\right)\right) + 4K\theta\eta_t + 4\eta_{tt} - 2\eta V_{\phi\phi} \\
+2V_\phi\left(\eta - 2\xi_t + \eta_\phi\right) + 2V\left(2\xi_t + \eta_\phi\right) - 2R'\xi - 2R\left(2\xi_t - \eta_\phi\right) = 0.
\end{gathered}$$

Now we impose the symmetry $[\xi = at, \eta = 1]$, $a \in \mathbb{R}$, which brings us to get the following restrictions on the potential, V, and the other quantities θ, R and h □

Grupos de Lie. Segunda Solución

Theorem

If KG eq is invariant under $[a,1]$, $a\in\mathbb{R}$, *then*

$$\phi=\frac{1}{a}t,\quad \theta=const,\quad R=const,\qquad V=C_1+C_2e^{\phi},\quad h=C_3e^{-\frac{1}{a}t}-C_4,$$

where $a,r,\phi_0,C_i\in\mathbb{R}$, *and from physical considerations we set*

$$\phi=\phi_0 t,\quad \theta=const,\quad R=r,\quad V=V_0,\qquad h=h_0e^{-\phi_0 t}=h_0e^{-\phi},$$

we obtain as solution (case FRW):

$$\phi=t,\quad \theta=const,\quad a=e^{a_1 t},\quad R=const,\quad V=V_0,\quad h=h_0e^{-t},\quad \rho=\rho_0e^{-t},$$

where $(1+\gamma)a_1=1$, that is: $a_1=\frac{1}{1+\gamma}$, with $\gamma\in(-1,1]$. So this solution is inflationary, $q=-1$, verifies the $T-$duality property $\left(a(t)\to a^{-1}(-t)\right)$ and has a constant potential, $V=V_0$.

Colineadores del Lagrangiano

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}e^{-\phi}\left[R+(\nabla\phi)^2-V-\frac{1}{12}\mathbf{H}^2\right]+\int d^4x\sqrt{-g}\mathscr{L}_{\text{matter}},$$

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}e^{-\phi}\left[R+(\nabla\phi)^2-V\right],$$

- $Q=(a,\phi)$ and $TQ=\left(a,\dot{a},\phi,\dot{\phi}\right)$
- flat FRW metric so $R=6\left(H^2+a''/a\right)$,
- the Lagrangian,

$$\mathscr{L}=e^{-\phi}\left(6a\dot{a}^2+a^3\left(\phi_t^2+V\right)-6a^2\dot{a}\phi_t\right),\qquad\left\|\partial_{\dot{q}_i\dot{q}_j}\mathscr{L}\right\|\neq 0.$$

Colineadores del Lagrangiano

- the lift vector X is now written as

$$X=\alpha\frac{\partial}{\partial a}+\beta\frac{\partial}{\partial\phi}+\frac{d\alpha}{dt}\frac{\partial}{\partial\dot{a}}+\frac{d\beta}{dt}\frac{\partial}{\partial\dot{\phi}},$$

If we calculate $L_X\mathscr{L}=0$, then,

$$\begin{aligned}\alpha-a\beta+2a\frac{\partial\alpha}{\partial a}-a^2\frac{\partial\beta}{\partial a}&=0,\\ 3\alpha-a\beta-6\frac{\partial\alpha}{\partial\phi}+2a\frac{\partial\beta}{\partial\phi}&=0,\\ -6\alpha+3a\beta-3a\frac{\partial\alpha}{\partial a}+6\frac{\partial\alpha}{\partial\phi}+a^2\frac{\partial\beta}{\partial a}-3a\frac{\partial\beta}{\partial\phi}&=0,\\ (3\alpha-a\beta)V+a\beta V'&=0,\end{aligned}$$

Colineadores del Lagrangiano

obtaining the following solutions:

1. Sol1
$$\alpha = C_1 a, \qquad \beta = 3C_1, \qquad V = V_0.$$
2. Sol2
$$\alpha = C_2 a^m e^{\phi/2}, \quad \beta = \frac{-3}{m-1} C_2 a^{m-1} e^{\phi/2}, \quad V = V_0 e^{m\phi}, \quad m = -(\sqrt{3}+1)/2,$$
so $V = V_0 e^{n\phi}$,
3. Sol3
$$\alpha = C_3 a + \left(C_1 a^{(\sqrt{3}-1)/2} + C_2 a^{(-\sqrt{3}-1)/2} \right) e^{\phi/2},$$
$$\beta = 3C_3 + \left(\left(\sqrt{3}+3\right) C_1 a^{(\sqrt{3}-3)/2} + \left(3-\sqrt{3}\right) C_2 a^{(-\sqrt{3}-3)/2} \right) e^{\phi/2},$$
$$V = 0.$$

three different cosmological scenarios with three potentials; constant, dynamical and vanishing.

Colineadores del Lagrangiano. Primera solución

The constant of motion generated by Sol2

1- $Q_2 = i_{\mathbf{X}_2}\theta_{\mathscr{L}}$ yields

$$Q_2 = i_{\mathbf{X}_2}\theta_{\mathscr{L}} = 2\sqrt{3}a^{(\sqrt{3}+1)/2}e^{-\phi/2}C_1\left(-\left(\sqrt{3}+3\right)\dot{a}+a\phi_t\right),$$

where the Cartan one form is given by $\theta_{\mathscr{L}} = \partial_{\dot{a}}\mathscr{L}da + \partial_{\dot{\phi}}\mathscr{L}d\phi$. If for example we set $Q_2 = 0$, then we get: $\phi = \phi_0 \ln a$.

2- From $L_X\mathscr{L} = 0$,

$$L_X\mathscr{L} = \alpha\partial_a\mathscr{L} + \dot{\alpha}\partial_{\dot{a}}\mathscr{L} + \beta\partial_a\mathscr{L} + \dot{\beta}\partial_{\dot{\phi}}\mathscr{L} = 0,$$

where $\partial_{\dot{a}}\mathscr{L} = p_a$, and, by taking into account the E-L Eqs., then $L_X\mathscr{L} = 0$ yields

$$\frac{d}{dt}\left(\alpha p_a + \beta p_\phi\right) = 0, \qquad Q = \alpha p_a + \beta p_\phi$$

so

$$(m-1)Q = 6a^{m+1}e^{-\phi/2}\left((2m+1)\dot{a} - ma\phi_t\right).$$

Colineadores del Lagrangiano. Primera solución

If we set $Q=0$, then we obtain

$$\phi=\frac{(2m+1)}{m}\ln a,\qquad m=-(\sqrt{3}+1)/2,\qquad \phi=\left(3-\sqrt{3}\right)\ln a,$$

3- As a final remark about invariant solution, it is possible to find an invariant solution

$$\frac{da}{\alpha}=\frac{d\phi}{\beta}\qquad\Longrightarrow\qquad \phi=\frac{3}{1-m}\ln a=\left(3-\sqrt{3}\right)\ln a,$$

which is the conserved quantity deduced previously

Colineadores del Lagrangiano. Primera solución

for $z=z(a,\phi)$, and $w=w(a,\phi)$, $i_{\mathbf{X}}(dz)=1$, $i_{\mathbf{X}}(dw)=0$, finding, for example (there are several solutions and not all of them works well), the following one:

$$w=\left(\sqrt{3}-3\right)\ln a+\phi, \qquad z=\frac{\sqrt{3}}{3}a^{\frac{1}{2}\left(3+\sqrt{3}\right)}e^{-\frac{1}{2}\phi},$$

and therefore

$$a=e^{\frac{\sqrt{3}}{6}w}\left(\frac{\sqrt{3}}{z}\right)^{-\frac{\sqrt{3}}{3}},$$

$$\phi=\frac{1}{2}\left(w\left(\sqrt{3}+1\right)+\left(\sqrt{3}-1\right)\ln 3+2\left(1-\sqrt{3}\right)\ln\left(\frac{1}{z}\right)\right).$$

Colineadores del Lagrangiano. Primera solución

By rewriting the Lagrangian in the coordinates (w,z) it yields:

$$\overline{\mathscr{L}}=-e^{-\frac{1}{2}w}\left(2\sqrt{3}\dot{z}\dot{w}+V_0e^{-\frac{1}{2}w\left(\sqrt{3}+2\right)}\right),$$

and therefore the new E-L Eqs. are:

$$\ddot{w}=\frac{\dot{w}^2}{2},$$
$$\ddot{z}=\frac{1}{4}\left(1+\sqrt{3}\right)V_0e^{-\frac{1}{2}w\left(\sqrt{3}+2\right)},$$

finding in this way that

$$w=-2\ln\left(-\frac{1}{2}\left(c_1t+c_2\right)\right),\qquad c_1,c_2,c_3,c_4\in\mathbb{R},$$

$$z=K\left(c_1t+c_2\right)^4\left(-\frac{1}{2}\left(c_1t+c_2\right)\right)^{\sqrt{3}}+c_3t+c_4,$$

Colineadores del Lagrangiano. Primera solución

Now we recover the solution in the original variables (a,ϕ)

$$a=\frac{\left(2\sqrt{3}\right)^{\sqrt{3}/3}}{\left(-c_1t-c_2\right)^{\frac{1}{3}\sqrt{3}}}\left(K\left(c_1t+c_2\right)^4\left(-\frac{1}{2}\left(c_1t+c_2\right)\right)^{\sqrt{3}}+c_3t+c_4\right)^{\frac{\sqrt{3}}{3}},$$

$$\phi=-\ln\left(-\frac{1}{2}\left(c_1t+c_2\right)\right)\left(\sqrt{3}+1\right)$$
$$+\left(1-\sqrt{3}\right)\ln\left(\frac{1}{K\left(c_1t+c_2\right)^4\left(-\frac{1}{2}\left(c_1t+c_2\right)\right)^{\sqrt{3}}+c_3t+c_4}\right)+C_1,$$

Solutions with physical meaning if $(c_i)_{i=1}^4\leq 0$. Solution collapses to SSS

Colineadores del Lagrangiano. Segunda solución

With regard to the first of the symmetries that is, $\alpha = a$, $\beta = 3$ and $V = V_0$,
1.- $Q = \alpha p_a + \beta p_\phi$, and therefore $-e^{-\phi} 6a^2 \dot{a} = Q$, finding in this way that

$$a = \frac{1}{2}\left(-4Q\int e^{\phi}dt + 8c_1\right)^{1/3},$$

but we are not able to obtain more information.
2.- An invariant solution

$$\frac{da}{\alpha} = \frac{d\phi}{\beta} \qquad \Longrightarrow \qquad \phi = 3\ln a,$$

Colineadores del Lagrangiano. Segunda solución

Now if we calculate the cv, $z = z(a, \phi)$, and $w = w(a, \phi)$, induced by the symmetry, we may find that:

$$w = -3\ln a, \qquad z = \ln a, \Longrightarrow a = e^z, \qquad \phi = w + 3z$$

. With these new variables the Lagrangian yields

$$\mathscr{L} = e^{-w}\left(\dot{w}^2 - 3\dot{z}^2 + V_0\right),$$

in such a way that the new EL equations are:

$$\ddot{z} = \dot{w}\dot{z},$$
$$2\ddot{w} = \dot{w}^2 + 3\dot{z}^2 - V_0,$$

finding that

$$z = c_1 + c_2 \int e^w dt,$$
$$w = w_0,$$

Calculating the inverse cv, we arrive at the following solution:

$$a = e^t, \qquad \phi = \phi_0 t,$$

thus, this particular solution is quite similar to the obtained one through the Lie group method with the symmetry $[a, 1]$.

Colineadores del Lagrangiano. Otras consideracines

If we take into account the $\mathbf{H}$ field

$$S=\frac{1}{2\kappa_D^2}\int d^D x\sqrt{-g}e^{-\phi}\left[R+(\nabla\phi)^2-V-\frac{1}{12}\mathbf{H}^2\right]$$

- $Q=(a,\phi,h)$ and $TQ=\left(a,\dot{a},\phi,\dot{\phi},h,\dot{h}\right)$
- Lagrangian, for flat FRW

$$\mathscr{L}=e^{-\phi}\left(12a\dot{a}^2+2a^3\left(\phi_t^2+V\right)-12a^2\dot{a}\phi_t-a^3e^{2\phi}h_t^2\right),$$

- field

$$\mathbf{X}=\alpha\frac{\partial}{\partial a}+\beta\frac{\partial}{\partial\phi}+\frac{d\alpha}{dt}\frac{\partial}{\partial\dot{a}}+\frac{d\beta}{dt}\frac{\partial}{\partial\dot{\phi}}+\gamma\frac{\partial}{\partial h}+\frac{d\gamma}{dt}\frac{\partial}{\partial\dot{h}},$$

Colineadores del Lagrangiano. Otras consideracines

- $L_X \mathscr{L} = 0$

$$
\begin{aligned}
\alpha - a\beta + 2a\frac{\partial \alpha}{\partial a} - a^2\frac{\partial \beta}{\partial a} &= 0, \\
3\alpha - a\beta - 6\frac{\partial \alpha}{\partial \phi} + 2a\frac{\partial \beta}{\partial \phi} &= 0, \\
-3\alpha - \beta a - 2a\frac{\partial \gamma}{\partial h} &= 0, \\
-6\alpha + 3a\beta + 6\frac{\partial \alpha}{\partial \phi} - 3a\frac{\partial \alpha}{\partial a} + a^2\frac{\partial \beta}{\partial a} - 3a\frac{\partial \beta}{\partial \phi} &= 0, \\
e^{-\phi}12\frac{\partial \alpha}{\partial h} - e^{-\phi}6a\frac{\partial \beta}{\partial h} - a^2 e^{\phi}\frac{\partial \gamma}{\partial a} &= 0, \\
-e^{-\phi}6\frac{\partial \alpha}{\partial h} + e^{-\phi}2a\frac{\partial \beta}{\partial h} - ae^{\phi}\frac{\partial \gamma}{\partial \phi} &= 0, \\
(3\alpha - a\beta)V + a\beta V' &= 0.
\end{aligned}
$$

Colineadores del Lagrangiano. Otras consideracines

Solutions

1.- $\alpha = C_1 a, \qquad \beta = 3C_1, \qquad \gamma = -3C_1 h + C_2, \qquad V = C_3.$

invariant solutions

$$\frac{da}{a} = \frac{d\phi}{3} \qquad \phi = 3\ln a$$
$$\frac{da}{a} = \frac{dh}{-3h} \qquad h = a^{-3}$$

2.- $\alpha = \beta = 0, \qquad \gamma = C_1, \qquad V = V.$

NOTA. Si consideramos el campo de materia $\mathscr{L}_{\text{matter}}$ entonces $V = V(a,\phi)$ más complicado

Soluciones Autosimilares

Estudiamos los siguientes modelos

- ${}^{\text{eff}}T_{\mu\nu} = {}^{(\phi)}T_{\mu\nu}$
- ${}^{\text{eff}}T_{\mu\nu} = {}^{(\phi)}T_{\mu\nu} + {}^{(V)}T_{\mu\nu}$
- ${}^{\text{eff}}T_{\mu\nu} = {}^{(\phi)}T_{\mu\nu} + {}^{(\mathbf{H})}T_{\mu\nu}$
- ${}^{\text{eff}}T_{\mu\nu} = {}^{(\phi)}T_{\mu\nu} + {}^{(\mathbf{H})}T_{\mu\nu} + {}^{(\mathrm{V})}T_{\mu\nu}$
- ${}^{\text{eff}}T_{\mu\nu} = {}^{(m)}T_{\mu\nu} + {}^{(\phi)}T_{\mu\nu} + {}^{(\mathbf{H})}T_{\mu\nu}$
- ${}^{\text{eff}}T_{\mu\nu} = {}^{(m)}T_{\mu\nu} + {}^{(\phi)}T_{\mu\nu} + {}^{(V)}T_{\mu\nu}$
- ${}^{\text{eff}}T_{\mu\nu} = {}^{(m)}T_{\mu\nu} + {}^{(\phi)}T_{\mu\nu} + {}^{(\mathbf{H})}T_{\mu\nu} + {}^{(V)}T_{\mu\nu}$

Soluciones Autosimilares

Recall that the main physical quantities behaves as follow:

$$\rho = \rho_0 t^{-(\phi_0+2)}, \quad \phi = \phi_0 \ln t, \quad h = h_0 t^{-\phi_0} + h_1, \quad V = \Lambda e^{-\frac{2}{\phi_0}\phi},$$

Metrics

- Flat FRW

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} \left(dx^i\right)^2$$

- Bianchi II

$$ds^2 = -dt^2 + a^2 dx^2 + \left(b^2 + K^2 z^2 a^2\right) dy^2 + 2Ka^2 z dx dy + d^2 dz^2,$$

the scale factors must behave as: $a(t) = t^{a_1}$, $b(t) = t^{a_2}$, $d(t) = t^{a_3}$, such that $a_2 + a_3 = 1 + a_1$

Soluciones Autosimilares. Modelo 1

$$^{\mathrm{eff}}T_{\mu\nu} = {}^{(\phi)}T_{\mu\nu}$$

Modelo FRW

$$a(t) = t^{\sqrt{3}/3}, \quad \phi = \left(\sqrt{3} - 1\right) \ln|t|.$$

$q = \sqrt{3} - 1 > 0$, so the solution is not inflationary.

Note that $\phi_0 > 0$, thus the function e^{ϕ} is unbounded as $t \in \mathbb{R}^{+}$.

Modelo BII

1.- Self-similar condition (SSC)
$a_2 + a_3 = 1 + a_1,$

$$K = 0,$$
$$a_2 = \frac{1}{2}\left(1 + a_1 + \sqrt{-\left(a_1 + 1\right)\left(3a_1 - 1\right)}\right),$$
$$a_3 = \frac{1}{2}\left(1 + a_1 - \sqrt{-\left(a_1 + 1\right)\left(3a_1 - 1\right)}\right),$$
$$\phi_0 = 2a_1.$$

this solutions belongs to the BI
2.- we relax the hypothesis SCC solution belongs to the BI

Soluciones Autosimilares. Modelo 2

$$ {}^{\text{eff}}T_{\mu\nu} = {}^{(\phi)}T_{\mu\nu} + {}^{(V)}T_{\mu\nu} $$

Modelo FRW

$$ a = t^{a_1}, \quad \phi = \phi_0 \ln t, \quad V = \Lambda e^{-\frac{2}{\phi_0}\phi}, $$

where

$$ a_1 = \frac{\phi_0}{2-\phi_0}, \qquad \Lambda = \frac{\phi_0^2\left(\phi_0^2 + 2\phi_0 - 2\right)}{\left(\phi_0 - 2\right)^2}, $$

$q = 2\left(1/\phi_0 - 1\right)$ with $q < 0$ if $\phi_0 \in (1,2)$.

$\Lambda > 0$ if $\phi_0 \in (0.732, 2)$,

Modelo BII

- SS $a_2 + a_3 = 1 + a_1$

BII with

$$ a_2 = a_3 = \frac{1}{2}\left(a_1 + 1\right) \quad \phi_0 = \frac{6a_1 + 2}{5 + 3a_1}. $$

($a_1 > 0$, when $t \in \mathbb{R}^+$), thus the function e^{ϕ} is unbounded

- Without the SSC, then we get a FRW like solution,

Soluciones Autosimilares. Modelo 3

$$^{\mathrm{eff}}T_{\mu\nu} = {}^{(\phi)}T_{\mu\nu} + {}^{(\mathbf{H})}T_{\mu\nu}$$

No existen soluciones reales

Modelo FRW

$$a(t) = t^{1/3}, \quad \phi = \left(\frac{2}{3}\right)\ln|t|$$

$$h = it^{\mp\frac{2}{3}}, \quad h \in \mathbb{C}$$

unphysical solution

Modelo BII

- SS $a_2 + a_3 = 1 + a_1$

$$a_1 = 0, \quad a_2 = a_3 = \frac{1}{2} = \phi_0$$

$$h = i \in \mathbb{C}, \quad K = \frac{1}{2}i \in \mathbb{C},$$

- Without the SSC, we get a FRW solution,

Soluciones Autosimilares. Modelo 4

$$^{\mathrm{eff}}T_{\mu\nu} = {}^{(\phi)}T_{\mu\nu} + {}^{(\mathbf{H})}T_{\mu\nu} + {}^{(\mathrm{V})}T_{\mu\nu}$$

Modelo FRW

$$a = t^{1/3}, \qquad \phi = \phi_0 \ln|t|,$$

$$h_0 = \frac{2\sqrt{3-6\phi_0}}{3\phi_0}, \quad \Lambda = \phi_0\left(\phi_0 - \frac{2}{3}\right),$$

-$q = 2$, thus it is not inflationary.
-h is growing $\phi_0 < 0$,
-e^{ϕ} is bounded in this model.
- $V > 0 \Longleftrightarrow \phi_0 < 0$.

Modelo BII

- SS $a_2 + a_3 = 1 + a_1$

$$a_1 = 0, \quad a_2 = a_3 = \frac{1}{2}$$

Unphysical solution

- Without the SSC, we get the FRW solution,

Soluciones Autosimilares. Modelo 5

$$^{\mathrm{eff}}T_{\mu\nu} = \ ^{(m)}T_{\mu\nu} + \ ^{(\phi)}T_{\mu\nu} + \ ^{(\mathbf{H})}T_{\mu\nu}$$

Modelo FRW

$$a = t^{1/3}, \qquad \phi = (\gamma - 1)\ln|t|,$$

$$h_0 = \frac{\sqrt{18\gamma^2 - 36\gamma + 6}}{3(\gamma - 1)}, \quad \rho_0 = \frac{5}{3} - \gamma,$$

-$q = 2$, thus it is not inflationary.
-$h_0 \in \mathbb{R}$ iff $\gamma \in (-1, 0.183) = I$
-e^{ϕ} is bounded in this model.
-$\rho_0 > 0$.

Modelo BII

- SS $a_2 + a_3 = 1 + a_1$

$$a_1 = 0, \quad a_2 = a_3 = \frac{1}{2}$$

Unphysical solution

- Without the SSC, we get the FRW solution,

Soluciones Autosimilares. Modelo 6 FRW

$$^{\mathrm{eff}}T_{\mu\nu} = {}^{(m)}T_{\mu\nu} + {}^{(\phi)}T_{\mu\nu} + {}^{(V)}T_{\mu\nu}$$

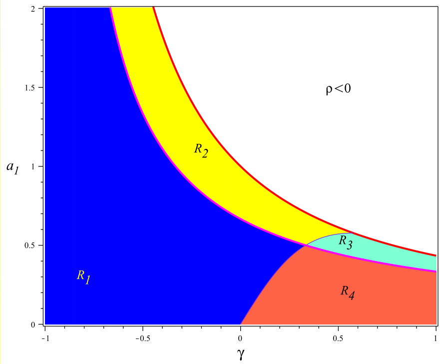
Modelo FRW

$$a = t^{a_1}, \quad \phi_0 = \phi_0(a_1, \gamma)$$
$$\Lambda = \Lambda(a_1, \gamma)$$
$$\rho_0 = \rho_0(a_1, \gamma)$$

we assume $\gamma \in (-1,1)$ and $a_1 \in (0,2)$

	ρ	ϕ	V
$\mathscr{R}_1$	>0	<0	<0
$\mathscr{R}_2$	>0	>0	>0
$\mathscr{R}_3$	>0	>0	<0
$\mathscr{R}_4$	>0	<0	>0

$$\mathscr{R} = \cup_{i=1}^{4} \mathscr{R}_i$$



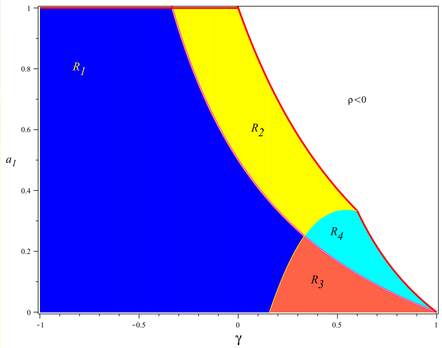
Soluciones Autosimilares. Modelo 6 BII

$$^{\mathrm{eff}}T_{\mu\nu} = \ ^{(m)}T_{\mu\nu} + \ ^{(\phi)}T_{\mu\nu} + \ ^{(V)}T_{\mu\nu}$$

Modelo BII solution type BII self-similar:

$$K^2 = \frac{1}{2}\left(a_1 - 1\right)\left(\gamma\left(2a_1+1\right)-1\right),$$
$$a_2 = a_3 = \frac{1}{2}\left(a_1+1\right),$$
$$\phi_0 = 2a_1\left(\gamma+1\right)+\gamma-1,$$
$$\rho_0 = \rho_0(a_1,\gamma)$$
$$\Lambda = \Lambda(a_1,\gamma)$$

$$\mathscr{R} = \cup_{i=1}^{4}\mathscr{R}_i$$



	ρ	ϕ	V
$\mathscr{R}_1$	>0	<0	<0
$\mathscr{R}_2$	>0	>0	>0
$\mathscr{R}_3$	>0	<0	>0
$\mathscr{R}_4$	>0	>0	<0

Soluciones Autosimilares. Modelo 7

$$^{\mathrm{eff}}T_{\mu\nu} = {}^{(m)}T_{\mu\nu} + {}^{(\phi)}T_{\mu\nu} + {}^{(\mathbf{H})}T_{\mu\nu} + {}^{(V)}T_{\mu\nu}$$

Modelo FRW=BII

$$a_1 = \frac{1}{3}, \quad \phi_0 = \gamma - 1, \quad h_0 = h_0$$
$$\Lambda = \Lambda(h_0, \gamma)$$
$$\rho_0 = \rho_0(h_0, \gamma)$$

- $a_1 = 1/3$ $q > 0$ $\mathbf{H}$-field
- assume $\gamma \in (-1,1)$ $h_0 \in (-2,2)$. $\mathscr{R}$
- $\phi_0 < 0$, for all $\gamma \in (-1,1)$, e^{ϕ} is bounded at late times