POLYMERIC QUANTIZATION OF HIGHER-DERIVATIVE MODELS

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PLAN OF THE TALK

- **BEYOND THE STANDARD MODEL: QUANTUM GRAVITY AND HIGHER-ORDER OPERATORS**
- **HIGHER-ORDER TIME DERIVATIVE MODELS: STABILITY PROBLEM**
- **THE PAIS-UHLENBECK MODEL**
- **POLYMERIC QUANTIZATION**
- **SUMMARY**

THE STANDARD MODEL

The standard model has been extremely successful in describing the interactions of particles, it is a precise and predictive theory.

However, there are good arguments to believe that the SM as an effective theory with a more fundamental theory ruling at higher energies.

From the QFT community,

- **Off shell effects lead to complicated proof of unitarity of the S-matrix and gauge invariance.**
- **A huge number of diagrams (for example at tree level for gluon scattering $g + g \to g + g + g + g$ one needs 220 diagrams).**
- **Free parameters and a fine-tuning problem in the Higgs sector.**

From the gravity community,

- **Gravity is not included in the quantum framework.**
- **Divergent integrals and divergent series which may be solved if the gravitational field comes into play.**

If one consider the standard model is an effective theory then one can ask what are the new ingredients coming from a more fundamental theory.

For example,

- Extra dimensions and non-locality.
- Supersymmetry.
- Lorentz invariance violation.
- Fundamental discreteness.
- Higher-order operators.

In this work we want to explore some possible consequences of fundamental discreteness in a quantum field theory with higher-order operators.

HIGHER-ORDER OPERATORS

Higher-order operators have been used in several contexts. In general they are good candidates to describe new physics.

- **Initially they were suggested to soften the divergencies that appear in quantum field theory** [B. Podolsky, Phys. Rev. 62, 68 (1942)].

- **They have been obtained in semmiclassical aproaches to loop quantum gravity** [R. Gambini and J. Pullin, Phys. Rev. D 59, 124021 (1999); J. Alfaro, H. A. Morales-Tecotl and L. F. Urrutia, Phys. Rev. Lett. 84, 2318 (2000), H. Sahlmann and T. Thiemann, Class. Quant. Grav. 23, 867 (2006).

- **They arise as a consequence of the nonlocal action in string theory** [D. A. Eliezer and R. P. Woodard, Nucl. Phys. B 325, 389 (1989).].

- **In a different but equivalent scenario they have been studied by Lee and Wick in order to prove unitarity of negative metric theories** [T. D. Lee and G. C. Wick, Nucl. Phys. B 9, 209 (1969); T. D. Lee and G. C. Wick, Phys. Rev. D 2, 1033 (1970).

For the special case of including higher-order time derivatives in Newtonian systems, STABILITY is automatically put in question as pointed out by Ostrogradski many years ago (1850).

In what follows, first we review well-known studies on theories containing higher-order time derivatives. We want to show,

1) How does the instability appear?.

2) What happens after quantization?.

And second, try to contribute to answer the question,

It is possible to avoid the instability by incorporating fundamental discreteness in the higher-order theory? [P. Cumsille, CMR, S. Ossandon and C. Reyes, Int. J. Mod. Phys. A 31, no. 09, 1650040 (2016).

OSTROGRADSKI FORMULATION

In the Ostrogradski formalism [M. Ostrogradski, Mem. Acad. St. -Pétersbourg VI, 385 (1850)], one considers a Lagrangian with k-th temporal derivatives,

$$L = L(q, \dot{q}, \ddot{q} \cdots, q^{(k)}), \tag{1}$$

where $q^{(k)}$ is the k-th temporal derivative of q.

Consider the action

$$S[q] = \int_{t_1}^{t_2} dt \; L(q, \dot{q}, \ldots). \tag{2}$$

The usual variation of the action with respect to q produces the extended Euler-Lagrange equation

$$\sum_{i=0}^{k} \left(-\frac{d}{dt} \right)^i \frac{\partial L}{\partial q^{(i)}} = 0, \tag{3}$$

plus a boundary terms

$$\left[\sum_{i=0}^{k-1} P_i \dot{q}^{(i)}\right]_{t_1}^{t_2}, \tag{4}$$

where

$$P_i(t) = \frac{\partial L}{\partial q^{(i+1)}} + \sum_{j=1}^{k-i-1} \left(-\frac{d}{dt}\right)^j \frac{\partial L}{\partial q^{(j+1)}}, \quad i = 0, \ldots, (k-1), \tag{5}$$

are the canonical conjugate momenta of $q^{(i)}$ satisfying

$$\{q^{(i)}, P_j\} = \delta_{ij}. \tag{6}$$

The Hamiltonian is

$$H = \sum_{i=0}^{k-1} P_i \dot{q}^{(i)} - L(q, \dot{q}, \cdots, q^{(k)}), \tag{7}$$

where we have assumed non-degeneracy, that is one can solve $q^{(k)}$ as a function of q_0, ..., q_{k-1}, P_{k-1}.

The above expression is linear in some momenta and coordinates making the Hamiltonian unbounded from below. This is the classical Ostrogradski instability.

For example, consider the Lagrangian

$$L(q,\dot{q},\ddot{q}) = \frac{\dot{q}^2}{2} + \frac{aq^2}{2} + gq\ddot{q}^2. \tag{8}$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0, \tag{9}$$

$$\ddot{q} = aq + 2g\frac{\partial^2}{\partial t^2}(q\ddot{q}). \tag{10}$$

Momenta are

$$P_q = \frac{\partial L}{\partial \dot{q}} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \ddot{q}}\right) = \dot{q} - 2g\frac{\partial}{\partial t}(q\ddot{q}), \qquad P_{\dot{q}} = \frac{\partial L}{\partial \ddot{q}} = 2gq\ddot{q}. \tag{11}$$

And the Hamiltonian

$$H \quad = \quad P_q\dot{q} + P_{\dot{q}}\ddot{q} - L = -\frac{\dot{q}^2}{2} - \frac{aq^2}{2} + P_q\dot{q} + \frac{P_{\dot{q}}^2}{4gq} \tag{12}$$

is function of $q, \dot{q}, P_q, P_{\dot{q}}$.

Upon quantization?

With a redefinition of the vacuum state one can solve the problem of stability but the price to pay is to have ghost states that may spoil unitarity evolution [A. Pais and G. E. Uhlenbeck, Phys. Rev. 79, 145 (1950).].

Lee, Wick and Cutkosky studied these theories and showed that unitarity can be preserved by demanding asymptotic states to have positive metric [T. D. Lee and G. C. Wick, Nucl. Phys. B 9, 209 (1969); T. D. Lee, G. C. Wick, Phys. Rev. D2, 1033-1048 (1970); R. E. Cutkosky, P. V. Landshoff, D. I. Olive, J. C. Polkinghorne, Nucl. Phys. B12, 281-300 (1969)] .

Recently using the ideas of Lee-Wick-Cutkosky it has been proved that unitarity can be preserved at one-loop level in the Myers y Pospelov QED model [M. Maniatis and CMR, Phys. Rev. D 89, 056009 (2014)] .

THE PAIS UHLENBECK MODEL

Let us consider a quantum mechanical model called the Pais-Uhlenbeck [A. Pais and G. E. Uhlenbeck, Phys. Rev. 79, 145 (1950)] . The Lagrangian is basically the harmonic oscillator plus a higher-order time derivative term,

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 - \frac{g}{2} \ddot{x}^2 \ , \tag{13}$$

where g is a small coupling. The equation of motion is

$$-kx - m\ddot{x} - gx^{(4)} = 0 \ . \tag{14}$$

where $x^{(4)}$ is a fourth time-derivative. We follow the canonical formalism for higher-order time derivative theories, and momenta are,

$$\pi = \frac{\partial L}{\partial \ddot{x}} = -g\ddot{x} \ , \tag{15}$$

$$p = \frac{\partial L}{\partial \dot{x}} - \dot{\pi} = m\dot{x} + gx^{(3)} \ , \tag{16}$$

The Hamiltonian is given by

$$H = p\dot{x} + \pi\ddot{x} - L \ , \tag{17}$$

which replacing Eqs. (13)**,** (15) **and** (16) **produces**

$$H = -\frac{\pi^2}{2g} - \frac{m\dot{x}^2}{2} + \dot{x}p + \frac{kx^2}{2} \ . \tag{18}$$

We define new canonical variables

$$x_1 = \frac{\omega_2^2 x - \pi/g}{\omega_2^2 - \omega_1^2} \ , \qquad x_2 = -\frac{\omega_1^2 x - \pi/g}{\omega_2^2 - \omega_1^2} \ , \tag{19}$$

and

$$p_1 = p - (m - g\omega_2^2)\dot{x} \ , \qquad p_2 = p - (m - g\omega_1^2)\dot{x} \ , \tag{20}$$

with

$$\omega_1 = \omega_0 \sqrt{\frac{1 - \sqrt{1-4\xi}}{2\xi}} \,, \qquad \omega_2 = \omega_0 \sqrt{\frac{1 + \sqrt{1-4\xi}}{2\xi}} \,, \tag{21}$$

where $g = \frac{m\xi}{\omega_0^2}$. They allow to define the creation a and annihilation variables

$$a_j = \sqrt{\frac{m'\omega_j}{2\hbar}}(x_j + \frac{i}{m'\omega_j}p_j) \,, \qquad j = 1, 2, \tag{22}$$

where $m' = \sqrt{m^2 - 4kg}$, such that the Hamiltonian reads

$$H = \frac{\hbar\omega_1}{2}(a_1 a_1^\dagger + a_1^\dagger a_1) - \frac{\hbar\omega_2}{2}(a_2 a_2^\dagger + a_2^\dagger a_2) \,. \tag{23}$$

We can quantize two decoupled harmonic oscillators, and the total energy is the difference of individual energies of the positive and negative parts.

Under the presence of interactions the negative part can grow indefinitely leading to the instability.

THE POLYMERIC REPRESENTATION

In quantum mechanics one finds technical difficulties due to the unboundness of the operators $\hat{x}$ and $\hat{p}$. Sometimes it is more convenient to work with the exponentiated versions $\hat{U}(\alpha)$ y $\hat{V}(\beta)$

$$\hat{U}(\alpha) = e^{i\alpha\hat{x}}, \qquad \hat{V}(\beta) = e^{i\beta\hat{p}/\hbar} \, , \tag{24}$$

which action are defined by multiplication

$$\hat{U}(\alpha)\psi(x) = e^{i\alpha x}\psi(x) \, , \tag{25}$$

and translation

$$\hat{V}(\beta)\psi(x) = \psi(x+\beta) \, , \tag{26}$$

for all the states $\psi(x)$ in the Hilbert space $L^2(\mathbb{R})$.

Both operators $\hat{U}(\alpha)$ and $\hat{V}(\beta)$ satisfy the Weyl-Heisenberg algebra

$$\begin{aligned}
\hat{U}(\alpha)\hat{U}(\alpha') &= \hat{U}(\alpha+\alpha')\ , \\
\hat{V}(\beta)\hat{V}(\beta') &= \hat{V}(\beta+\beta')\ , \\
\hat{U}(\alpha)\hat{V}(\beta) &= e^{-i\alpha\beta}\hat{V}(\beta)\hat{U}(\alpha)\ ,
\end{aligned} \tag{27}$$

where α and β have dimensions of L^{-1} and L respectively.

According to the Stone Von Neuman theorem if U y V are weakly continuous in the parameters any representation of the algebra is unitarity equivalent to $L^2(\mathbb{R})$.

To construct the polymeric Hilbert space we start with a graph given by a numerable set of points on the real line denoted by $\gamma = \{x_j : j \in \mathbb{N}\}$.

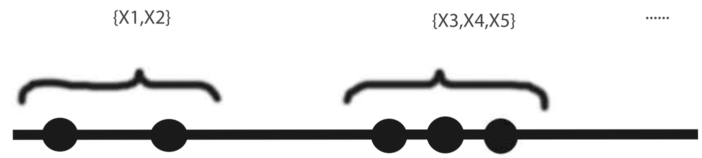

Figura 1: Two graphs $\gamma_1 = (x_1, x_2)$ and $\gamma_2 = (x_3, x_4, x_5)$

We define the function $f(x)$ associated to the graph γ as

$$f(x) = \begin{cases} f_j & x = x_j \\ 0 & x \notin \gamma \end{cases} \tag{28}$$

Denote by Cyl_γ the space of complex valued functions $f(k)$ of the type

$$f(k) = \sum_j f_j e^{-ix_j k}, \tag{29}$$

that satisfy the relation $\sum_j |f_j|^2 x_j^{2n} < \infty$ for all $n = 0, 1, 2, \ldots$

Consider all possible graphs and denote by Cyl the infinite dimensional vector space of functions on $\mathbb{R}$ which are cylindrical with respect to some graph.

We introduce an inner product in the space Cyl given by

$$\langle e^{-ikx_i} | e^{-ikx_j} \rangle = \delta_{x_i, x_j}. \tag{30}$$

Now, the polymeric Hilbert space H_{poly} is the Couchy completion of Cyl.

One can show that the polymeric representation has a Hilbert space H_{poly} which is

1) nonseparable

2) and has an intrinsic discreeteness (which eventually leads to an nonequivalent representation).

Using the scalar product of Eq. (30) **one can show that $\hat{V}(\beta)$ is not weakly continuos in the parameter β.**

Hence, there is no adjoint operator $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ satisfying (24)**. This operator $\hat{p}$ is not well defined in H_{poly}.**

However, for practical purposes one can approximate the operator $\hat{p}$ in the Hamiltonians by the expression

$$\hat{p} = -\frac{i\hbar}{\mu_0}\left(\hat{V}(\mu_0/2) - \hat{V}(-\mu_0/2)\right) \ , \tag{31}$$

where μ_0 is a length scale associated to the space discreetness.

The construction has been applied in the more complete program of loop quantum cosmology [M. Martin-Benito, G. A. Mena Marugan and T. Pawlowski, Phys. Rev. D 78, 064008 (2008); M. Bojowald, Living Rev. Rel. 11, 4 (2008).] .

POLYMERIC QUANTIZATION OF THE P-U MODEL

We can write the Hamiltonian as

$$H = H_1 - H_2 \ , \tag{32}$$

with

$$H_1 = \frac{1}{2}k_1 x_1^2 + \frac{1}{2m'}p_1^2 \ , \qquad H_2 = \frac{1}{2}k_2 x_2^2 + \frac{1}{2m'}p_2^2 \ , \tag{33}$$

where $k_j = m'\omega_j^2$ with $j = 1, 2$.

Now we follow the work given in [A. Ashtekar, S. Fairhurst and J. L. Willis, Class. Quant. Grav. 20, 1031 (2003).] , for the quantization of one harmonic oscillator.

We have the Schrödinger equation for the wave function $\psi(x_1, x_2) \in H_{poly}$

$$\left(\frac{1}{2}k_1\hat{x}_1^2 + \frac{1}{2m'}\hat{p}_1^2 - \frac{1}{2}k_2\hat{x}_2^2 - \frac{1}{2m'}\hat{p}_2^2\right)\psi(x_1, x_2) = E\psi(x_1, x_2) \ , \tag{34}$$

with E the total energy of the system.

We consider the ansatz $\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$ and the notation for the oscillator

$$\psi_J(k) = \sum_\ell \psi_J(x_{J,\ell})e^{-ix_{J,\ell}k}, \tag{35}$$

with $J = 1, 2$.

The equation for the coefficients turns to be

$$\psi_J(x_{J,\ell} + \mu_J) + \psi_j(x_{J,\ell} - \mu_J)\left(2 - \frac{2E_J\mu_J^2}{\hbar\omega_J d_J^2} + \frac{\mu_J^2 x_{J,\ell}^2}{d_J^4}\right)\psi_J(x_{J,\ell}) \ , \tag{36}$$

where $E = E_1 - E_2$ and $d_J^2 = \frac{\hbar}{m'\omega_J}$.

Without loss of generality, we base both graphs on the point $x_0 = 0$. With $x_{J,\ell} = \ell\mu_J$, and $\psi_J(k_J) = \sum_{\ell=-\infty}^{\infty} \psi_J(x_{J,\ell}) e^{-ik_J\ell\mu_J}$ we multiply Eq. (36) by $e^{-ik\ell\mu_J}$ and take the sum over ℓ to arrive at

$$2\cos(k_J\mu_J)\psi_J(k_J) = 2\left(1 - \frac{E_J\mu_J^2}{\hbar\omega_J d_J^2}\right)\psi_J(k_J) - \frac{\mu_J^2}{d_J^4}\psi_J''(k_J)\ , \tag{37}$$

where $k_J \in (-\frac{\pi}{\mu_J}, \frac{\pi}{\mu_J})$. We have

$$\psi_J''(k_J) + 2d_J^2\left(\frac{E_J}{\hbar\omega_J} + \frac{d_J^2}{\mu_J^2}(\cos(k_J\mu_J) - 1)\right)\psi_J(k_J) = 0\ . \tag{38}$$

Finally, with the change of variables $z_J = \frac{k_J\mu_J + \pi}{2} \in (0, \pi)$ we write

$$\psi_J''(z_J) + \Big(a_J - 2q_J\cos(2z_J)\Big)\psi_J(z_J) = 0\ , \tag{39}$$

where $q_J = 4\lambda_J^{-4}$, $a_J = \frac{8}{\lambda_J^4}\left(\frac{\lambda_J^2 E_J}{\hbar\omega_J} - 1\right)$ and for the same size of the lattice spacing μ_0, we have the dimensionless parameter $\lambda_J = \mu_0/d_J$.

This is the Mathieu equation with a well known asymptotic expansion for the eigenvalues when λ is small or large.

Since the first oscillator is a perturbation we will consider small λ_1 with energy

$$E_{1,n} = \frac{\hbar\omega_1}{2}\left((2n+1) - \frac{1}{16}(2n^2+2n+1)\lambda_1^2\right) + \mathcal{O}(\lambda_1^4)\ . \tag{40}$$

For the second oscillator we have two alternatives.

The first one is to consider small λ_2, that is to say $\mu_0 \ll d_2$ having energy

$$E_{2,m} = \frac{\hbar\omega_2}{2}\left((2m+1) - \frac{1}{16}(2m^2+2m+1)\lambda_2^2\right) + \mathcal{O}(\lambda_2^4)\ . \tag{41}$$

The second is a large λ_2 leading to

$$E_{2,m} \quad = \quad \frac{\hbar\omega_2}{\lambda_2^2} + \hbar\omega_2\frac{\lambda_2^2}{8}m^2 + \mathcal{O}(\lambda_2^{-6})\ . \tag{42}$$

We consider a cutoff in the energy of the fists oscillator, since

1) Above $n = 10^7$ the polymeric effects are in the range of observability [A. Ashtekar, S. Fairhurst and J. L. Willis, Class. Quant. Grav. 20, 1031 (2003).]

2) Required for a consistent renormalization [A. Corichi, T. Vukasinac and J. A. Zapata, Class. Quant. Grav. 24, 1495 (2007); A. Corichi, T. Vukasinac and J. A. Zapata, Phys. Rev. D 76, 044016 (2007).]

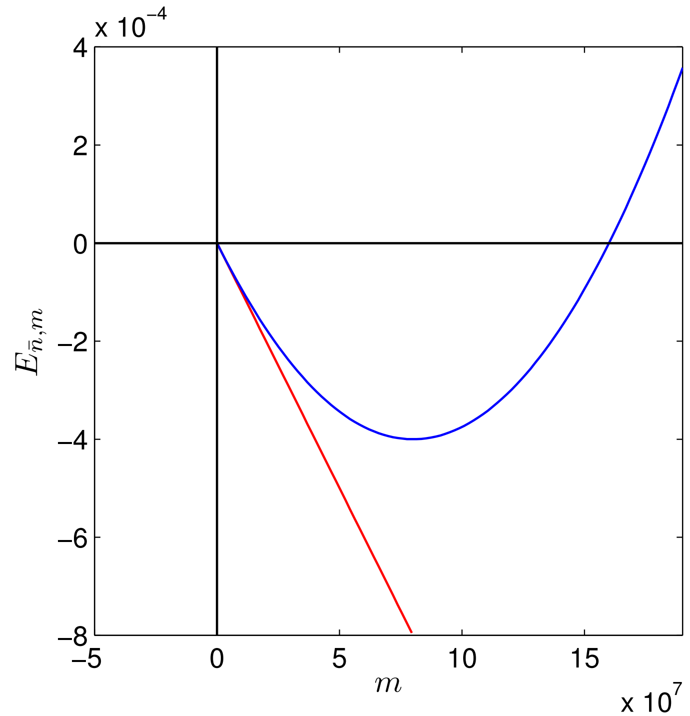

Figura 2: Comparison of the P-U energies as a function of m for fixed $\bar{n} = 10^7$ in the Schrodinger (red) and polymeric representations (blue)

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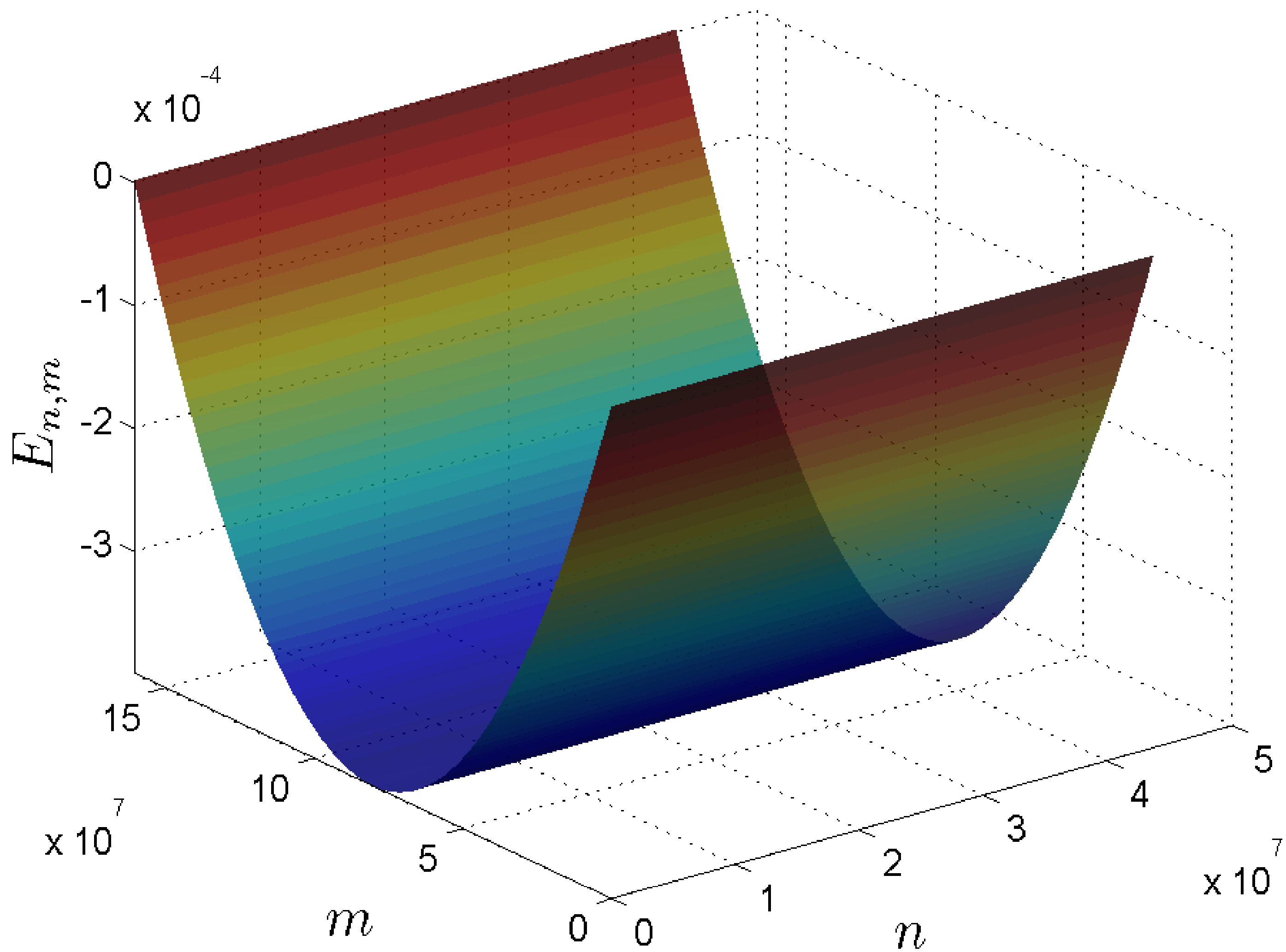

Figura 3: The polymeric P-U energy $E_{n,m}$ as a function of n and m.

CONCLUSIONS

- We have polymeric quantized the P-U model seen as two decoupled harmonic oscillators.

- We have shown new regions where the model has a well bounded Hamiltonian.

- The idea that a level of discreetness may regulate singularities in quantum field theory has been considered using the polymeric approach for higher-order theories. It should be viewed as a first step in order to really solve the problem of stability.

- In future work we plan to study a more direct analysis considering the set of original variables without the canonical transformation and with interactions.