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Spherically symmetric Einstein-aether perfect fluid models

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Contenido

Introduction

- Einstein-aether theory [Donnelly & Jacobson 2010, Jacobson & Mattingly 2001] consists of GR coupled, at second derivative order, to a dynamical timelike unit vector field, the aether.
- The aether spontaneously breaks Lorentz invariance by picking out a preferred frame at each point in spacetime while maintaining local rotational symmetry (breaking only the boost sector of the Lorentz symmetry).
- In the IR limit of (extended) Horava gravity [Horava 2009] [a candidate UV completion in the consistent non-projectable extension of Horava-Lifschitz gravity], the aether vector is assumed to be hypersurface-orthogonal; hence every hypersurface-orthogonal Einstein-aether solution is a Horava solution (most of the solutions studied).
- The relationship between Einstein-aether theory and Horava gravity is further clarified in [Jacobson 2013], where it is shown how Horava gravity can formally be obtained from Einstein-aether theory in the limit that the twist coupling constant goes to infinity.
- Cosmological models in aether theories of gravity are currently of interest. In particular, it is of importance to study inhomogeneous cosmologies, in both GR and alternative gravitational theories, partially motivated by current cosmological observations.

Einstein-aether model

In an Einstein-aether model there will be additional terms in the FE which include [Donnelly & Jacobson 2010, Jacobson & Mattingly 2001]:

- The effects on the geometry from the anisotropy and inhomogeneities (e.g., the curvature) of the spherically symmetric models under consideration.
- The Einstein FE are generalised by the contribution of an additional stress tensor, S_{ab}, for the aether field which depends on the dimensionless parameters of the aether model (e.g., "the c_i"). In GR, all of the $c_i = 0$. To study the effects of matter, we could perhaps assume the corresponding GR values (or close to them) in the first instance.
- When the phenomenology of theories with a preferred frame is studied, it is generally assumed that this frame coincides, at least roughly, with the cosmological rest frame defined by the Hubble expansion of the universe. In particular, in an isotropic and spatially homogeneous Friedmann universe the aether field will be aligned with the (natural preferred CMB rest frame) cosmic frame and is thus related to the expansion rate of the universe. In principle, the preferred frame determined by the aether can be different from (i.e., tilted with respect to) the CMB rest frame in spherically symmetric models. This adds additional terms to the aether stress tensor S_{ab}, which can be characterized by a hyperbolic tilt angle, $v(t)$, measuring the boost of the aether relative to the (perfect fluid) CMB rest frame [Carruthers & Jacobson 2011, Kanno & Soda 2006]. The tilt is expected to decay to the future in anisotropic but spatially homogeneous models [Coley & Hervik 2005, Coley, Hervik & Lim 2006].

We follow a similar approach to that in the resource paper [Coley, Lim & Leon 2008].
The metric is:

$$ds^2 = -N^2 dt^2 + ({e_1}^1)^{-2} dx^2 + ({e_2}^2)^{-2}(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \quad (1)$$

The Killing vector fields (KVF) are given by:

$$\partial_\varphi, \quad \cos\varphi\ \partial_\vartheta - \sin\varphi \cot\vartheta\ \partial_\varphi, \quad \sin\varphi\ \partial_\vartheta + \cos\varphi\cot\vartheta\ \partial_\varphi. \quad (2)$$

The frame vectors in coordinate form are:

$$\mathbf{e}_0 = N^{-1}\partial_t, \quad \mathbf{e}_1 = {e_1}^1 \partial_x, \quad \mathbf{e}_2 = {e_2}^2\partial_\vartheta, \quad \mathbf{e}_3 = {e_3}^3 \partial_\varphi, \quad (3)$$

where ${e_3}^3 = {e_2}^2 / \sin\vartheta$. N, ${e_1}^1$ and ${e_2}^2$ are functions of t and x.
This leads to the following restrictions on the kinematic variables:

$$\sigma_{\alpha\beta} = \mathrm{diag}(-2\sigma_+, \sigma_+, \sigma_+), \quad \omega_{\alpha\beta} = 0, \quad \dot{u}_\alpha = (\dot{u}_1, 0, 0), \quad (4)$$

where

$$\dot{u}_\alpha = u^\beta \nabla_\beta u_\alpha; \quad (5)$$

$$\dot{u}_1 = \mathbf{e}_1 \ln N; \quad (6)$$

on the spatial commutation functions:

$$a_\alpha = (a_1, a_2, 0), \quad n_{\alpha\beta} = \begin{pmatrix} 0 & 0 & n_{13} \\ 0 & 0 & 0 \\ n_{13} & 0 & 0 \end{pmatrix}, \tag{7}$$

where

$$a_1 = \mathbf{e}_1 \ln {e_2}^2, \quad a_2 = n_{13} = -\frac{1}{2}{e_2}^2 \cot\vartheta; \tag{8}$$

and on the matter components:

$$q_\alpha = (q_1, 0, 0), \quad \pi_{\alpha\beta} = \mathrm{diag}(-2\pi_+, \pi_+, \pi_+). \tag{9}$$

The frame rotation $\Omega_{\alpha\beta}$ is also zero.
Furthermore, n_{13} only appears in the equations together with $\mathbf{e}_2 n_{13}$ in the form of the Gauss curvature of the spheres

$$^2K := 2(\mathbf{e}_2 - 2n_{13})n_{13}, \tag{10}$$

which simplifies to

$$^2K = ({e_2}^2)^2. \tag{11}$$

Thus the dependence on ϑ is hidden in the equations. We will also use ^2K in place of ${e_2}^2$.
To simplify notation, we will write $^2K,\ \dot{u}_1,\ a_1$ as $K,\ \dot{u},\ a$. To summarize, the essential variables are $N, {e_1}^1,\ K,\ \theta,\ \sigma_+,\ a,\ \dot{u},\ \mu,\ q_1,\ p,\ \pi_+$.

The action for Einstein-aether theory [Jacobson 2007, Garfinkle & Jacobson 2011, Carroll & Lim 2004]:

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}R - K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d + \lambda\left(u^c u_c + 1\right) + \mathcal{L}_m\right], \tag{12}$$

where

$$K^{ab}{}_{cd} \equiv c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c + c_4 u^a u^b g_{cd}. \tag{13}$$

The field equations from varying (12) with respect to g^{ab}, u^a, and λ are given, respectively, by [Garfinkle, Eling & Jacobson 2007]:

$$G_{ab} = T^{TOT}_{ab} \tag{14}$$

$$\lambda u_b = \nabla_a J^a{}_b + c_4 \dot{u}_a \nabla_b u^a \tag{15}$$

$$u^a u_a = -1. \tag{16}$$

Here G_{ab} is the Einstein tensor of the metric g_{ab}. T^{TOT}_{ab} is the *total* energy momentum tensor, $T^{TOT}_{ab} = T_{ab} + T^{mat}_{ab}$, where T^{mat}_{ab} is the total contribution from all matter sources.

The quantities ${J^a}_b$, $\dot{u}_a$ and the aether stress-energy T_{ab} are given by

$$
\begin{aligned}
{J^a}_m &= -{K^{ab}}_{mn}\nabla_b u^n && \text{(17a)}\\
\dot{u}_a &= u^b\nabla_b u_a && \text{(17b)}\\
T_{ab} &= 2c_1(\nabla_a u^c\nabla_b u_c - \nabla^c u_a\nabla_c u_b)\\
&\quad -2[\nabla_c(u_{(a}{J^c}_{b)}) + \nabla_c(u^c J_{(ab)}) - \nabla_c(u_{(a}{J_{b)}}^c)] - 2c_4\dot{u}_a\dot{u}_b +\\
&\quad +2\lambda u_a u_b + g_{ab}\mathscr{L}_u. && \text{(17c)}
\end{aligned}
$$

where

$$\mathscr{L}_u \equiv -{K^{ab}}_{cd}\nabla_a u^c\nabla_b u^d, \tag{18}$$

is the Einstein-aether Lagrangian [Jacobson & Mattingly 2004].
Taking the contraction of (15) with u^b and with the induced metric $h^{bc} := g^{bc} + u^b u^c$ we obtain the equations

$$
\begin{aligned}
\lambda &= -u^b\nabla_a J^a_b - c_4\dot{u}_a\dot{u}^a, && \text{(19a)}\\
0 &= h^{bc}\nabla_a J^a_b + c_4 h^{bc}\dot{u}_a\nabla_b u^a. && \text{(19b)}
\end{aligned}
$$

[Coley, Lim & Leon 2008]:

$$\mathbf{e}_0({e_1}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+){e_1}^1, \tag{20a}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K, \tag{20b}$$

$$\mathbf{e}_0(\theta) - \mathbf{e}_1(\dot{u}) = -\tfrac{1}{3}\theta^2 - 6\sigma_+^2 + (\dot{u} - 2a)\dot{u} - \frac{1}{2}(\mu + 3p), \tag{20c}$$

$$\mathbf{e}_0(\sigma_+) - \frac{1}{3}\mathbf{e}_1(\dot{u} - a) = -\theta\sigma_+ - \frac{1}{3}(a + \dot{u})\dot{u} - \frac{1}{3}K + \pi_+, \tag{20d}$$

$$\mathbf{e}_0(a) = -\tfrac{1}{3}(\theta + 3\sigma_+)(a + \dot{u}) - \tfrac{1}{2}q_1, \tag{20e}$$

$$\mathbf{e}_0(\mu) + \mathbf{e}_1(q_1) = -\theta(\mu + p) + 2(a - \dot{u})q_1 - 6\sigma_+\pi_+, \tag{20f}$$

$$\mathbf{e}_0(q_1) + \mathbf{e}_1(p) = -\tfrac{2}{3}(2\theta - 3\sigma_+)q_1 - 2(3a - \dot{u} - \mathbf{e}_1)\pi_+ - \dot{u}(\mu + p), \tag{20g}$$

$$\mathbf{e}_1(\ln N) = \dot{u}, \tag{21a}$$

$$\mathbf{e}_1(\ln K) = 2a, \tag{21b}$$

$$\mu = 3H^2 - 3\sigma_+^2 + K - 3a^2 + 2\mathbf{e}_1(a), \tag{21c}$$

$$q_1 = -6a\sigma_+ + \tfrac{2}{3}\mathbf{e}_1(\theta + 3\sigma_+), \tag{21d}$$

The aether equation (19b) becomes

$$(c_1 - c_4)\mathbf{e}_0(\dot{u}) = -\tfrac{2}{3}(c_1 - c_4)(\theta + 3\sigma_+)\dot{u} + 6(c_1 + c_3)a\sigma_+ \tfrac{1}{3}(c_1 + 3c_2 + c_3)\mathbf{e}_1(\theta) - 2(c_1 + c_3)\mathbf{e}_1(\sigma_+). \tag{22}$$

The matter sources

The aether energy components

$$
\begin{aligned}
\mu &= -c_\theta \theta^2 - 6c_\sigma \sigma_+^2 - c_a(\dot{u} + 2\mathbf{e}_1 - 4a)\dot{u}, &(23a)\\
p &= -\tfrac{1}{3}c_a\dot{u}^2 - 6c_\sigma\sigma_+^2 + c_\theta(2\mathbf{e}_0 + \theta)\theta, &(23b)\\
q &= \tfrac{4}{3}c_a(\theta + 3\sigma_+)\dot{u} + 2c_a\mathbf{e}_0(\dot{u}), &(23c)\\
\pi &= -\tfrac{2}{3}c_a\dot{u}^2 + 2c_\sigma(\mathbf{e}_0 + \theta)\sigma_+. &(23d)
\end{aligned}
$$

where

$$c_\theta = c_2 + (c_1 + c_3)/3,\ c_\sigma = c_1 + c_3,\ c_\omega = c_1 - c_3,\ c_a = c_4 - c_1,$$

The energy momentum-tensor components for the tilted fluid:

$$
\begin{aligned}
\mu &= \frac{G_+}{1-v^2}\hat{\mu}, \qquad p = \frac{(\gamma-1)(1-v^2) + \frac{1}{3}\gamma v^2}{1-v^2}\hat{\mu} &(24a)\\
q_1 &= \frac{\gamma\hat{\mu}}{1-v^2}v, \qquad \pi_+ = -\frac{1}{3}\frac{\gamma\hat{\mu}}{1-v^2}v^2, &(24b)
\end{aligned}
$$

where $G_\pm = 1 \pm (\gamma - 1)v^2$, v is the tilt parameter. We choose a linear equation of state for the perfect fluid:

$$\hat{p} = (\gamma - 1)\hat{\mu}, \tag{25}$$

where γ is a constant satisfying $1 \leq \gamma < 2$.

Special case: $\dot{u} = 0$

$$\mathbf{e}_0({e_1}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+){e_1}^1, \tag{26a}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K, \tag{26b}$$

$$\mathbf{e}_0(\theta) = -\frac{1}{3}\theta^2 + \frac{6(2c_\sigma - 1)\sigma_+^2}{3c_\theta + 1} + \frac{\left(2 - 3\gamma + (\gamma - 2)v^2\right)\hat{\mu}}{2(3c_\theta + 1)(1 - v^2)}, \tag{26c}$$

$$\mathbf{e}_0(\sigma_+) = \frac{1}{2}\sigma_+^2 - \theta\sigma_+ + \frac{(3c_\theta + 1)\theta^2}{18(2c_\sigma - 1)} - \frac{a^2}{2(2c_\sigma - 1)} + \frac{K}{2(2c_\sigma - 1)} + \frac{\left((1 + \gamma)v^2 - 1\right)\hat{\mu}}{6(2c_\sigma - 1)(1 - v^2)}, \tag{26d}$$

$$\mathbf{e}_0(a) = -\frac{1}{3}a(\theta + 3\sigma_+) + \frac{\gamma v\hat{\mu}}{2(1 - v^2)}, \tag{26e}$$

$$\mathbf{e}_0(\hat{\mu}) - \frac{\mathbf{e}_1(\hat{\mu})v(2 - \gamma)}{G_-} - \frac{\gamma\hat{\mu}\mathbf{e}_1(v)}{G_-} = -\frac{2\gamma\hat{\mu}v^2\sigma_+}{G_-} + \frac{\gamma\left(v^2 - 3\right)\hat{\mu}\theta}{3G_-} - 2\frac{\gamma\hat{\mu}va}{G_-}, \tag{26f}$$

$$\mathbf{e}_0(v) - \frac{\mathbf{e}_1(\hat{\mu})\left(1 - v^2\right)^2(\gamma - 1)}{\gamma\hat{\mu}G_-} - \frac{v(2 - \gamma)\mathbf{e}_1(v)}{G_-} = \frac{2v\left(1 - v^2\right)\sigma_+}{G_-} + \frac{v\left(1 - v^2\right)(3\gamma - 4)\theta}{3G_-} + \frac{2v^2\left(1 - v^2\right)(\gamma - 1)a}{G_-}. \tag{26g}$$

Constraints $(3c_\theta + 2c_\sigma \neq 0)$

$$\mathbf{e}_1(\ln K) = 2a, \tag{27a}$$

$$\mathbf{e}_1(a) = \frac{G_+\hat{\mu}}{2(1-v^2)} - \frac{1}{6}(3c_\theta + 1)\,\theta^2 - \frac{3}{2}(2c_\sigma - 1)\,\sigma_+^2 - \frac{K}{2} + \frac{3a^2}{2} \tag{27b}$$

$$\mathbf{e}_1(\theta) = -\frac{3\gamma c_\sigma v\hat{\mu}}{(3c_\theta + 2c_\sigma)(1-v^2)}, \tag{27c}$$

$$\mathbf{e}_1(\sigma_+) = 3a\sigma_+ - \frac{3\gamma c_\theta v\hat{\mu}}{2(3c_\theta + 2c_\sigma)(1-v^2)}. \tag{27d}$$

Normalized variables

We introduce the normalized variables (for $\dot{u} \neq 0$ using the β-normalization):

$$\left\{\mathcal{N}^{-1}, E_1^1, \mathcal{Q}, \Sigma, \mathcal{A}, \dot{U}\right\} = \left\{N^{-1}, e_1^1, \frac{\theta}{3}, \sigma_+, a, \dot{u}\right\} / \beta$$

$$\{\Omega, \Omega_k, \mathcal{K}, \mathcal{S}\} = \left\{\hat{\mu}, -\tfrac{1}{2}{}^3R, K, {}^3S_+\right\} / (3\beta^2),$$

where $\beta = \frac{1}{3}(\theta + 3\sigma_+)$ and

$${}^3R = 4\mathbf{e}_1 a - 6a^2 + 2K,\ {}^3S_+ = -\tfrac{1}{3}\mathbf{e}_1 a + \tfrac{1}{3}K.$$

By definition $\mathcal{Q} + \Sigma = 1$. In the above we assume that $\beta \neq 0$. In general the variables are unbounded. However, physically $\Omega \geq 0$, and if the expansion and the shear are both positive, then $0 \leq \mathcal{Q} \leq 1$.

We also introduce the normalized differential operators,

$$\boldsymbol{\partial}_\alpha := \frac{\mathbf{e}_\alpha}{\beta} \quad ; \text{ where } \quad \alpha = 0, 1.$$

Moreover, we define q and r analogous to the usual volume deceleration parameter and “Hubble spatial gradient” as follows

$$\boldsymbol{\partial}_0\beta := -(1+q)\beta, \quad \boldsymbol{\partial}_1\beta := -r\beta. \tag{28}$$

Normalized equations

$$\partial_0 E_1^1 = (q + 3\Sigma) E_1^1, \tag{29a}$$

$$\partial_0 \mathcal{K} = 2q\mathcal{K}, \tag{29b}$$

$$\partial_0 \mathcal{Q} = \mathcal{Q}(1 + q - \mathcal{Q}) + \frac{2(2c_\sigma - 1)\Sigma^2}{3c_\theta + 1} + \frac{\Omega\left((\gamma-2)v^2 - 3\gamma + 2\right)}{2(3c_\theta + 1)(1 - v^2)}, \tag{29c}$$

$$\partial_0 \Sigma = \Sigma(1 + q - 3\mathcal{Q}) + \frac{1}{2}\Sigma^2 + \frac{3\mathcal{K}}{2(2c_\sigma - 1)} - \frac{\mathcal{A}^2}{2(2c_\sigma - 1)} + \frac{(3c_\theta + 1)\mathcal{Q}^2}{2(2c_\sigma - 1)} + \\ + \frac{\Omega\left((\gamma+1)v^2 - 1\right)}{2(2c_\sigma - 1)(1 - v^2)}, \tag{29d}$$

$$\partial_0 \mathcal{A} = q\mathcal{A} + \frac{3\gamma v \Omega}{2(1 - v^2)}, \tag{29e}$$

$$G\_\partial_0 \Omega + (\gamma - 2) v \partial_1 \Omega - \gamma \Omega \partial_1 v = -2\gamma \mathcal{A} v \Omega + 2q\Omega\left(1 - (\gamma - 1)v^2\right) + \\ -3\gamma \mathcal{Q}\left(1 - v^2\right)\Omega + 2(\gamma - 2) r v \Omega + 2\Omega\left((1 - 2\gamma)v^2 + 1\right), \tag{29f}$$

$$\gamma \Omega G\_\partial_0 v - (\gamma - 1)\left(1 - v^2\right)^2 \partial_1 \Omega + \gamma \Omega (\gamma - 2) v \partial_1 v = 2\gamma \Omega (\gamma - 1)\mathcal{A}\left(1 - v^2\right)v^2 + \\ -3\gamma \Omega (\gamma - 2)\mathcal{Q}\left(1 - v^2\right)v - 2\Omega(\gamma - 1) r \left(1 - v^2\right)^2 + 2\gamma \Omega \left(1 - v^2\right) v. \tag{29g}$$

Restrictions

$$\partial_1 \mathcal{N}^{-1} = r\mathcal{N}^{-1}, \tag{30a}$$

$$\partial_1 \mathcal{K} = 2(r+\mathcal{A})\mathcal{K}, \tag{30b}$$

$$\partial_1 \mathcal{A} = -\frac{3}{2}\mathcal{K} + r\mathcal{A} + \frac{3}{2}\mathcal{A}^2 - \frac{3}{2}(3c_\theta + 2c_\sigma)\mathcal{Q}^2 + 3(2c_\sigma - 1)\mathcal{Q} + \frac{3\Omega\left((\gamma-1)v^2+1\right)}{2(1-v^2)} - 3c_\sigma + \frac{3}{2}, \tag{30c}$$

$$\partial_1 \mathcal{Q} = r\mathcal{Q} - \frac{3\gamma c_\sigma v\Omega}{(3c_\theta + 2c_\sigma)(1-v^2)}, \tag{30d}$$

$$\partial_1 \Sigma = r\Sigma + 3\mathcal{A}\Sigma - \frac{9\gamma c_\theta v\Omega}{2(3c_\theta + 2c_\sigma)(1-v^2)}. \tag{30e}$$

$$q = -\frac{3\mathcal{K}}{2(2c_\sigma - 1)} + \frac{\mathcal{A}^2}{2(2c_\sigma - 1)} - \frac{(3c_\theta + 2c_\sigma)(3c_\theta + 8c_\sigma - 3)\mathcal{Q}^2}{2(3c_\theta + 1)(2c_\sigma - 1)} + \frac{4(3c_\theta + 2c_\sigma)\mathcal{Q}}{3c_\theta + 1} + - \frac{\Omega\left(v^2(3(\gamma+1)c_\theta + 2(\gamma-2)c_\sigma + 3) - 6\gamma c_\sigma - 3c_\theta + 4c_\sigma + 3\gamma - 3\right)}{2(3c_\theta + 1)(2c_\sigma - 1)(1-v^2)} + \frac{1 - 9c_\theta - 8c_\sigma}{2(3c_\theta + 1)}, \tag{31}$$

$$r = -3\mathcal{A}(1-\mathcal{Q}) + \frac{3\gamma v\Omega}{2(1-v^2)}. \tag{32}$$

$$\partial_0 \mathcal{K} = 2q\mathcal{K}, \tag{33a}$$

$$\partial_0 \mathcal{Q} = \mathcal{Q}(1+q-\mathcal{Q}) + 2(2c_\sigma-1)(1-\mathcal{Q})^2 - \frac{1}{2}\Omega, \tag{33b}$$

$$\partial_0 \mathcal{A} = q\mathcal{A}, \tag{33c}$$

$$\partial_0 \Omega = (2q-3\mathcal{Q}+2)\Omega, \tag{33d}$$

$$\partial_1 \mathcal{K} = 2(r+\mathcal{A})\mathcal{K}, \tag{34a}$$

$$\partial_1 \mathcal{A} = -\frac{3}{2}\mathcal{K} + r\mathcal{A} + \frac{3}{2}\mathcal{A}^2 - 3c_\sigma \mathcal{Q}^2 + \frac{3}{2}(2c_\sigma-1)(2\mathcal{Q}-1) + \frac{3}{2}\Omega, \tag{34b}$$

$$\partial_1 \mathcal{Q} = r\mathcal{Q}, \tag{34c}$$

$$\tag{34d}$$

$$q = \frac{1}{2(2c_\sigma-1)}\Big\{-3\mathcal{K} + \mathcal{A}^2 - 2c_\sigma(8c_\sigma-3)\mathcal{Q}^2 + 16c_\sigma(2c_\sigma-1)\mathcal{Q} + 2c_\sigma\Omega + (1-8c_\sigma)(2c_\sigma-1)\Big\}, \quad r = -3\mathcal{A}(1-\mathcal{Q}). \tag{35}$$

We also have that $\partial_0 := \mathcal{N}^{-1}\partial_t, \quad \partial_1 := E_1^1\partial_x$. [The only remaining freedom is the coordinate rescalings $t\to f(t)$ and $x\to g(x)$].

$Q=1 \implies \Sigma=0 \implies r=0$, whence $\boldsymbol{\partial}_1\mathscr{N}^{-1}=0$, and we can rescale time so that $N=1$ and $\boldsymbol{\partial}_0 := \partial_\tau$, where τ is essentially logarithmic time.

$$\boldsymbol{\partial}_\tau E_1^1 = qE_1^1, \tag{36a}$$

$$\boldsymbol{\partial}_\tau \mathscr{K} = 2q\mathscr{K}, \tag{36b}$$

$$\boldsymbol{\partial}_\tau \mathscr{A} = q\mathscr{A}, \tag{36c}$$

$$\boldsymbol{\partial}_\tau \Omega = (2q-3\gamma+2)\Omega, \tag{36d}$$

subject to the restrictions:

$$\boldsymbol{\partial}_1 \mathscr{K} = 2\mathscr{A}\mathscr{K}, \tag{37a}$$

$$\boldsymbol{\partial}_1 \mathscr{A} = -\frac{3}{2}\mathscr{K} + \frac{3}{2}\mathscr{A}^2 + \frac{3}{2}(\Omega-1), \tag{37b}$$

$$\tag{37c}$$

where q is defined by:

$$q = \frac{1}{2}\Omega(3\gamma-2), \tag{38}$$

and where

$$0 = -3\mathscr{K} + \mathscr{A}^2 + \Omega - 1. \tag{39}$$

Exact solution

Define $\mathscr{D}=\mathscr{A}^2-3\mathscr{K}$, so that $\partial_\tau\mathscr{D}=2q\mathscr{D}$, subject to the restrictions: $\partial_\eta\mathscr{D}=3\left(\mathscr{D}+\Omega-1\right)$, where $\partial_\eta\equiv\partial_1/\mathscr{A}$. Since $0=\mathscr{D}+\Omega-1$ we then obtain $\partial_\eta\mathscr{D}=0$, which implies $\partial_\eta\Omega=0$. We then obtain

$$\mathscr{D}=\frac{1}{e^{-3\gamma\tau-c_1+2\tau}+1},\; \Omega=\frac{e^{2\tau}}{e^{3\gamma\tau+c_1}+e^{2\tau}},\; q=\frac{(3\gamma-2)e^{2\tau}}{2\left(e^{3\gamma\tau+c_1}+e^{2\tau}\right)},$$

$$\mathscr{A}=\frac{c_2(\eta)e^{\frac{3\gamma\tau}{2}}}{\sqrt{e^{3\gamma\tau+c_1}+e^{2\tau}}},\; \mathscr{K}=\frac{e^{3\gamma\tau}\left(c_2(\eta)^2-e^{c_1}\right)}{3\left(e^{3\gamma\tau+c_1}+e^{2\tau}\right)}.$$

$$\left.\begin{array}{r}\partial_\eta\mathscr{K}=2\mathscr{K},\\ \mathscr{A}\partial_\eta\mathscr{A}=-\frac{3}{2}\mathscr{K}+\frac{3}{2}\mathscr{A}^2+\frac{3}{2}(\Omega-1)\end{array}\right\}\implies e^{c_1}+c_2(\eta)\left(c_2'(\eta)-c_2(\eta)\right)=0, c_2(\eta)\neq 0, \tag{40}$$

$$\implies c_2(\eta)=\pm\sqrt{e^{c_1}-e^{2c_2+2\eta}}. \tag{41}$$

The above equations admit the solutions

$$E_1^1=\frac{c_3(\eta)e^{\frac{3\gamma\tau}{2}}}{\sqrt{e^{3\gamma\tau+c_1}+e^{2\tau}}},\mathscr{K}=-\frac{e^{3\gamma\tau+2c_2+2\eta}}{3\left(e^{3\gamma\tau+c_1}+e^{2\tau}\right)},\mathscr{A}=\pm\frac{\sqrt{e^{c_1}-e^{2(c_2+\eta)}}e^{\frac{3\gamma\tau}{2}}}{\sqrt{e^{3\gamma\tau+c_1}+e^{2\tau}}}.$$

Special dust model

The terms $\mathscr{K}, \mathscr{A}^2$, only appear in the evolution equations for $\partial_0\Omega, \partial_0\mathscr{Q}$ via (through q) the combination $\mathscr{D} \equiv \mathscr{A}^2 - 3\mathscr{K}$. Hence (assuming $c_\sigma \neq 0$) we have the reduced (closed) evolution system:

$$\partial_0\mathscr{D} = 2q\mathscr{D}, \tag{42a}$$

$$\partial_0\mathscr{Q} = \mathscr{Q}(1+q-\mathscr{Q}) + 2(2c_\sigma - 1)(1-\mathscr{Q})^2 - \frac{1}{2}\Omega, \tag{42b}$$

$$\partial_0\Omega = (2q - 3\mathscr{Q} + 2)\Omega, \tag{42c}$$

where

$$q = \frac{1}{2(2c_\sigma - 1)}\left\{\mathscr{D} - 2c_\sigma(8c_\sigma - 3)\mathscr{Q}^2 + 16c_\sigma(2c_\sigma - 1)\mathscr{Q} + 2c_\sigma\Omega + (1-8c_\sigma)(2c_\sigma - 1)\right\}. \tag{43}$$

Assuming $\mathscr{A} \neq 0$, we can define the new spatial derivative $\partial_\eta \equiv \mathscr{A}^{-1}\partial_1$, whence the spatial derivatives become:

$$\partial_\eta\mathscr{D} = 3\left(-2c_\sigma(1-\mathscr{Q})^2 + (\mathscr{D}-1)(-1+2\mathscr{Q}) + \Omega\right), \tag{44a}$$

$$\partial_\eta\mathscr{Q} = -3(1-\mathscr{Q})\mathscr{Q}. \tag{44b}$$

The commutator equation is given by

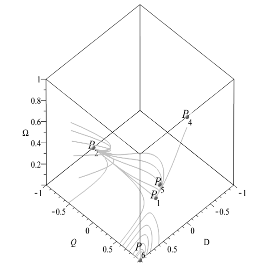
$$\left[\partial_\eta, \partial_\tau\right] = \mathscr{A}^{-1}\left(q\partial_0 + [\partial_1, \partial_0]\right). \tag{45}$$

Label	$(\mathscr{D},\mathscr{Q},\Omega)$	Eigenvalues
P_1	$\left(0,1+\frac{\sqrt{1-2c_\sigma}}{2c_\sigma}-\frac{1}{2c_\sigma},0\right)$	$\frac{4c_\sigma+3\sqrt{1-2c_\sigma}-3}{c_\sigma},\frac{8c_\sigma+3\sqrt{1-2c_\sigma}-3}{2c_\sigma},\frac{3\left(2c_\sigma+\sqrt{1-2c_\sigma}-1\right)}{2c_\sigma}$
P_2	$\left(0,1-\frac{\sqrt{1-2c_\sigma}}{2c_\sigma}-\frac{1}{2c_\sigma},0\right)$	$\frac{4c_\sigma-3\sqrt{1-2c_\sigma}-3}{c_\sigma},-\frac{-8c_\sigma+3\sqrt{1-2c_\sigma}+3}{2c_\sigma},-\frac{3\left(-2c_\sigma+\sqrt{1-2c_\sigma}+1\right)}{2c_\sigma}$
P_3	$\left(0,\frac{4(2c_\sigma-1)}{8c_\sigma-3},0\right)$	$-\frac{3}{8c_\sigma-3},-\frac{32c_\sigma-15}{2(8c_\sigma-3)},-1$
P_4	$(0,1,1)$	$1,1,-\frac{3}{2}$
P_5	$(1-2c_\sigma,\frac{2}{3},\frac{8c_\sigma}{9})$	$1,\frac{-3+\sqrt{9-48c_\sigma}}{6},\frac{-3-\sqrt{9-48c_\sigma}}{6}$
P_6	$(1,1,0)$	$-1,-1-\sqrt{\frac{2c_\sigma}{2c_\sigma-1}},-1+\sqrt{\frac{2c_\sigma}{2c_\sigma-1}}$
P_7	$\left(\frac{3(2c_\sigma-1)(8c_\sigma-3)}{(4c_\sigma-3)^2},\frac{2(2c_\sigma-1)}{4c_\sigma-3},0\right)$	$-\frac{4c_\sigma}{4c_\sigma-3},-\frac{3}{4c_\sigma-3},-\frac{8c_\sigma-3}{4c_\sigma-3}$

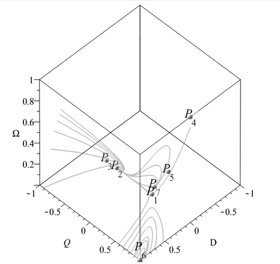
1. Point P_1 exists (i.e., with $-1 \leq Q \leq 1$) for $c_\sigma < 0$ or $0 < c_\sigma \leq \frac{1}{2}$. It is a source for $c_\sigma < 0$ or $0 < c_\sigma < \frac{3}{8}$; a saddle for $\frac{3}{8} < c_\sigma < \frac{1}{2}$. (The equilibrium points are non-hyperbolic for other values of the parameter c_σ).
2. Point P_2 exists for $\frac{3}{8} \leq c_\sigma \leq \frac{1}{2}$. It is a sink for $\frac{3}{8} \leq c_\sigma < \frac{15}{32}$; a saddle for $\frac{15}{32} < c_\sigma < \frac{1}{2}$.
3. Point P_3 exists for $c_\sigma \geq \frac{7}{16}$. It is a sink for $c_\sigma > \frac{15}{32}$; a saddle for $\frac{7}{16} \leq c_\sigma < \frac{15}{32}$.
4. The [FLRW] point P_4 always exists and it is a saddle.
5. The point P_5 always exists and it is a saddle for $c_\sigma \neq 0$.
6. The point P_6 always exists. It is sink for $c_\sigma < \frac{1}{2}$ [two complex conjugate eigenvalues with negative real part for $0 < c_\sigma < \frac{1}{2}$]. It is a saddle for $c_\sigma > \frac{1}{2}$. For $c_\sigma = \frac{1}{2}$ it is a saddle too.
7. The point P_7 exists for $c_\sigma \leq \frac{5}{8}$. It is a source for $\frac{3}{8} < c_\sigma \leq \frac{5}{8}$. Non-hyperbolic for $c_\sigma \in \{0, \frac{3}{8}\}$. Saddle otherwise.

Summary of sinks: P_6 for $c_\sigma < \frac{1}{2}$, P_2 for $\frac{3}{8} \leq c_\sigma < \frac{15}{32}$, P_3 for $c_\sigma > \frac{15}{32}$. In all cases $\Omega \to 0$ to the future. For P_2 and P_3, $\mathscr{D} \to 0$, but for P_6, $\mathscr{D} \to 1$ ($\mathscr{Q} \to 1$) and the shear goes to zero at late times (for small c_σ).

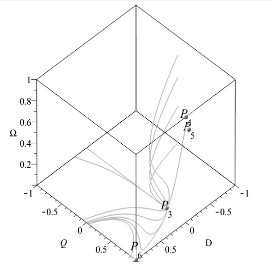
Phase space of the system (42)



For $c_\sigma = \frac{3}{8}$ the sinks are P_2 and P_6. P_7 and P_1 coincide; they are non-hyperbolic and behave as the sources.



For $c_\sigma = 0{,}45 \in \left(\frac{3}{8}, \frac{15}{32}\right)$ the sinks are P_2 and P_6, and P_7 is the source.



For $c_\sigma = 0{,}7$ the sink is P_3.

The metric has the form

$$ds^2 = -N(t)^2dt^2 + \ell^2(t)dx^2 + \ell^2(t)f^2(x)(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \qquad f(x) = \sin x,\ x,\ \sinh x, \quad (46)$$

y ${e_1}^1 = \ell^{-1}(t)$ and ${e_2}^2 = \ell^{-1}(t)f^{-1}(x)$. $\sigma_+ = \frac{1}{3}\mathbf{e}_0 \ln({e_1}^1/{e_2}^2)$, $a = -\frac{\partial_x f(x)}{f(x)\ell(t)}$.
$N = N(t) \implies \dot{u} = 0$. We usually choose $N = 1$. The Hubble scalar $H = \mathbf{e}_0 \ln \ell(t)$ is also a function of t.
In normalized coordinates are characterized by $\Sigma = 0$ $(Q = 1)$, $\dot{U} = 0$, $v = 0$, and we can use the remaining coordinate freedom to set $N = 1$ (where $\beta = \frac{\dot{\ell}}{N\ell}$). We recall that $\boldsymbol{\partial}_1 := E_1^1\partial_x$ and $\boldsymbol{\partial}_0 E_1^1 = qE_1^1$. We use the remaining spatial freedom to simplify $f(x)$ as in equation (**??**) for the FLRW metric as above, where $(\partial_x f(x))^2 \equiv 1 - \kappa f^2$, $\partial_x\partial_x f(x) \equiv -\kappa f$, and so we obtain: [1]

$$\mathcal{K} = \frac{N^2}{3\dot{\ell}^2 f^2}, \qquad \mathcal{A} = -\frac{N\partial_x f}{\dot{\ell} f}, \quad (47)$$

and hence

$$\mathcal{A}^2 - 3\mathcal{K} = -\frac{\kappa N^2}{\dot{\ell}^2}, \qquad \boldsymbol{\partial}_1\mathcal{A} = \frac{3}{N}\mathcal{K}. \quad (48)$$

[1] Flat FLRW power law models: In the flat case $\kappa = 0$, $f(x) = x$ (and $\tau = \ln(\ell)$). At the equilibrium points we have that $\ell = t^p$.

The spatially homogeneous spherically symmetric models (that has 4 Killing vectors, the fourth being ∂_x) are the so-called Kantowski-Sachs models. The metric (1) simplifies to

$$ds^2 = -N(t)^2 dt^2 + ({e_1}^1(t))^{-2} dx^2 + ({e_2}^2(t))^{-2}(d\vartheta^2 + \sin^2\vartheta d\varphi^2); \tag{49}$$

i.e., N, ${e_1}^1$ and ${e_2}^2$ are now independent of x. The spatial derivative terms $\mathbf{e}_1(\ )$ vanish and as a result $a = 0 = \dot{u}$. Since $\dot{u} = 0$, N is a positive function of t which under a time rescaling can be set to one. This metric choice forces the fluid to be non-tilted ($v = 0$) [assuming $\mu > 0, \gamma > 0$].

$$\mathbf{e}_0({e_1}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+){e_1}^1 \tag{50a}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K \tag{50b}$$

$$\mathbf{e}_0(\theta) = -\frac{\theta^2}{3} + \frac{6(2c_\sigma - 1)\sigma_+^2}{3c_\theta + 1} + \frac{(2-3\gamma)\hat{\mu}}{2(3c_\theta + 1)}, \tag{50c}$$

$$\mathbf{e}_0(\sigma_+) = -\frac{(3c_\theta + 1)\theta^2}{9(2c_\sigma - 1)} - \theta\sigma_+ - \sigma_+^2 + \frac{\hat{\mu}}{3(2c_\sigma - 1)}, \tag{50d}$$

$$\mathbf{e}_0(\hat{\mu}) = -\gamma\theta\hat{\mu} \tag{50e}$$

with the constraint

$$K + \frac{(3c_\theta + 1)\theta^2}{3} = \hat{\mu} - 3(2c_\sigma - 1)\sigma_+^2. \tag{51}$$

Normalized variables

We choose the following normalized variables (which are bounded for $1-2c_\sigma \geq 0$; note that we do not use the $\beta-$normalization for convenience here):

$$x=\frac{\sqrt{\hat{\mu}}}{D}, y=\frac{\sqrt{3}\sigma_+}{D}, z=\frac{\sqrt{K}}{D}, Q=\frac{\theta}{\sqrt{3}D} \tag{52}$$

where

$$D=\sqrt{K+\frac{\theta^2}{3}}, \tag{53}$$

and the new time variable $f' \equiv \frac{1}{D}\mathbf{e}_0(f)$.
The variables (52) are related through the constraints

$$-3c_\theta Q^2+x^2-(2c_\sigma-1)y^2=1, \tag{54a}$$

$$Q^2+z^2=1, \tag{54b}$$

which are preserved by the 4D system. From the equations (54) it follows that Q and z are bounded in the intervals $Q\in[-1,1]$, $z\in[0,1]$ (for expanding universes $Q\geq 0$). However, since $1-2c_\sigma$ is not necessarily non-negative it follows that x and y are unbounded, unless $1-2c_\sigma \geq 0$.

$$y' = \frac{\sqrt{3}Q^2}{3-6c_\sigma} + \frac{\sqrt{3}Qy\left(c_\theta\left((3\gamma-2)Q^2-4\right)+\gamma-2\right)}{6c_\theta+2} + \frac{\sqrt{3}(\gamma-2)Qy^3(2c_\sigma-1)}{6c_\theta+2} + \\ + \frac{1}{\sqrt{3}(2c_\sigma-1)} - \frac{\left(Q^2-1\right)y^2}{\sqrt{3}} \tag{55a}$$

$$Q' = \frac{\sqrt{3}(2-3\gamma)}{18c_\theta+6} + \frac{\sqrt{3}(3\gamma-2)Q^4 c_\theta}{6c_\theta+2} - \frac{\sqrt{3}(3\gamma-2)Q^2(3c_\theta-1)}{18c_\theta+6} + \\ + \frac{\sqrt{3}(\gamma-2)\left(Q^2-1\right)y^2(2c_\sigma-1)}{6c_\theta+2} + \frac{(1-Q^2)Qy}{\sqrt{3}} \tag{55b}$$

Special case.

Let us assume

$$3c_\theta \equiv c_1 + 3c_2 + c_3 = 0 \tag{56}$$

and define $c^2 \equiv 1 - 2c_\sigma = 1 - 2(c_1 + c_3) \geq 0$. This choice leads to a compact phase space. With these special values of the c's, the evolution equations for Kantowski-Sachs models simplify and the constraint becomes

$$K + \frac{\theta^2}{3} = \hat{\mu} + 3c^2\sigma_+^2. \tag{57}$$

This choice leads to a compact phase space.

Special case

The following normalized variable

$$y_1 = \frac{\sqrt{3}c\sigma_+}{D} \tag{58}$$

is chosen for convenience, whence the variables are related through the constraints

$$x^2 + y_1^2 = 1, \tag{59a}$$
$$Q^2 + z^2 = 1. \tag{59b}$$

With these special values of the c's, the evolution equations for Kantowski-Sachs models simplify and the constraint becomes

$$K + \frac{\theta^2}{3} = \hat{\mu} + 3c^2\sigma_+^2. \tag{60}$$

The following normalized variable

$$y_1 = \frac{\sqrt{3}c\sigma_+}{D} \tag{61}$$

is chosen for convenience, whence the variables are related through the constraints

$$x^2 + y_1^2 = 1, \tag{62a}$$
$$Q^2 + z^2 = 1. \tag{62b}$$

Thus, the phase space is compact with $x \in [-1, 1]$, $y_1 \in [-1, 1]$ and $Q \in [-1, 1]$, $z \in [0, 1]$ (for expanding universes $Q \geq 0$).

Reduced dynamical system

The system for (y_1, Q) reduces to

$$y_1' = -\frac{\left(y_1^2-1\right)\left(3c(\gamma-2)Qy_1+2Q^2-2\right)}{2\sqrt{3}c}, \tag{63a}$$

$$Q' = -\frac{\left(Q^2-1\right)\left(c\left(-3\gamma+3(\gamma-2)y_1^2+2\right)+2Qy_1\right)}{2\sqrt{3}c} \tag{63b}$$

Since the evolution equations are invariant under the transformation $y_1 \to -y_1$ and $c \to -c$, without loss of generality we can assume $c > 0$. Scaling the time derivative by the positive factor $2\sqrt{3}c$, we then obtain:

$$y_1' = -\left(y_1^2-1\right)\left(3c(\gamma-2)Qy_1+2Q^2-2\right), \tag{64a}$$

$$Q' = -\left(Q^2-1\right)\left(c\left(-3\gamma+3(\gamma-2)y_1^2+2\right)+2Qy_1\right) \tag{64b}$$

Equilibrium points of the system (64) and their eigenvalues.

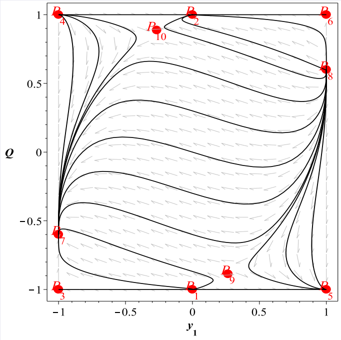
Label	Coordinates: (y_1, Q)	Eigenvalues
P_1	$(0,-1)$	$3c(2-\gamma), 2c(2-3\gamma)$
P_2	$(0,1)$	$-3c(2-\gamma), -2c(2-3\gamma)$
P_3	$(-1,-1)$	$-6c(2-\gamma), 4(1-2c)$
P_4	$(-1,1)$	$6c(2-\gamma), 4(1+2c)$
P_5	$(1,-1)$	$-6c(2-\gamma), -4(1+2c)$
P_6	$(1,1)$	$6c(2-\gamma), -4(1-2c)$
P_7	$(-1,-2c)$	$2(4c^2-1), 4\left[c^2(3\gamma-2)-1\right]$
P_8	$(1,2c)$	$-2(4c^2-1), -4\left[c^2(3\gamma-2)-1\right]$
P_9	$\left(\frac{c(2-3\gamma)}{d}, -\frac{2}{d}\right)$	$\frac{c(-e-3\gamma+6)}{d}, \frac{c(e-3\gamma+6)}{d}$
P_{10}	$\left(-\frac{c(2-3\gamma)}{d}, \frac{2}{d}\right)$	$\frac{c(-e+3\gamma-6)}{d}, \frac{c(e+3\gamma-6)}{d}$

$d = \sqrt{3(\gamma-2)(3\gamma-2)c^2+4}$ and $e \equiv \sqrt{3}\sqrt{2-\gamma}\sqrt{8c^2(2-3\gamma)^2-27\gamma+22}$

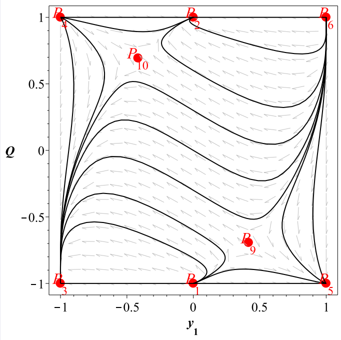
1. The equilibrium point P_1 is a source for $c > 0, 0 \leq \gamma < \frac{2}{3}$, and a saddle for $\frac{2}{3} < \gamma \leq 2$. Non-hyperbolic for $\gamma = \frac{2}{3}$ or $\gamma = 2$.
2. The equilibrium point P_2 is a sink for $c > 0, 0 \leq \gamma < \frac{2}{3}$, and a saddle for $\frac{2}{3} < \gamma < 2$. Non-hyperbolic for $\gamma = \frac{2}{3}$ or $\gamma = 2$.
3. The equilibrium point P_3 is a sink for $0 \leq \gamma < 2, c > \frac{1}{2}$, and non-hyperbolic for $c = \frac{1}{2}$ or $\gamma = 2$. Saddle otherwise.
4. The equilibrium point P_4 is a source for $c > 0, 0 \leq \gamma < 2$. Non-hyperbolic for $\gamma = 2$.
5. The equilibrium point P_5 is a sink for $c > 0, 0 \leq \gamma < 2$. Non-hyperbolic for $\gamma = 2$.
6. The equilibrium point P_6 is a source for $0 \leq \gamma < 2, c > \frac{1}{2}$. A saddle for $0 \leq \gamma < 2, 0 < c < \frac{1}{2}$. Non-hyperbolic for $\gamma = 2$ or $c = \frac{1}{2}$.
7. The equilibrium point P_7 exist for $0 \leq \gamma \leq 2, 0 < c \leq \frac{1}{2}$. It is a sink for $0 \leq \gamma \leq 2, 0 < c < \frac{1}{2}$. Non-hyperbolic otherwise.
8. The equilibrium point P_8 exists for $0 \leq \gamma \leq 2, 0 < c \leq \frac{1}{2}$. It is a source for $0 \leq \gamma \leq 2, 0 < c < \frac{1}{2}$. Non-hyperbolic otherwise.
9. The equilibrium point P_9 exists for $0 < c \leq \frac{1}{2}, 0 \leq \gamma \leq \frac{2}{3}$, or $0 < c \leq \frac{1}{2}, \gamma = 2$, or $c > \frac{1}{2}, 0 \leq \gamma \leq \frac{2}{3}$. It is a saddle for $0 \leq \gamma < \frac{2}{3}, c > 0$. Non-hyperbolic for $\gamma = \frac{2}{3}$ or $\gamma = 2$.
10. The equilibrium point P_{10} exists for $0 < c \leq \frac{1}{2}, 0 \leq \gamma \leq \frac{2}{3}$, or $0 < c \leq \frac{1}{2}, \gamma = 2$, or $c > \frac{1}{2}, 0 \leq \gamma \leq \frac{2}{3}$. It is a saddle for $0 \leq \gamma < \frac{2}{3}, c > 0$. Non-hyperbolic for $\gamma = \frac{2}{3}$ or $\gamma = 2$.

In the case $c_\sigma < \frac{1}{2}$ (i.e., $c > 0$), when $\gamma < \frac{2}{3}$, P_2 is the unique shear-free, zero curvature (FLRW) inflationary future attractor, and for $\frac{3}{8} < c_\sigma < \frac{1}{2}$ (i.e., $0 < c < \frac{1}{2}$) and $0 \leq \gamma < 2$ the sources and sinks are, respectively, P_4 & P_8 and P_5 & P_7. All of these sources and sinks have maximal shearing and all, except P_7, have zero curvature; the sink P_7 does not have zero curvature. For $c_\sigma < \frac{3}{8}$ (i.e., $c > \frac{1}{2}$) the points P_7 & P_8 do not exist, and the sources and sinks with maximal shearing are P_4 and P_5, respectively.

Phase space of the system (64).

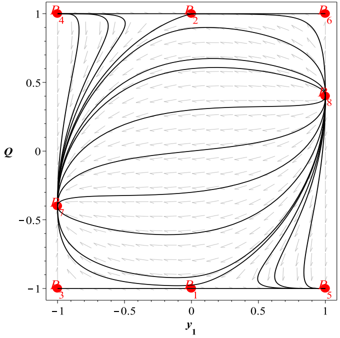


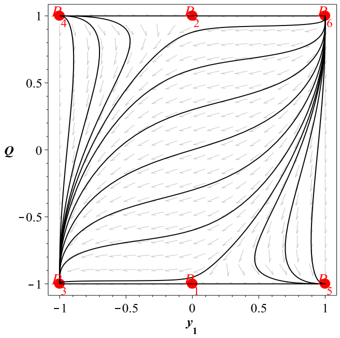
For $\gamma = 0$ and $c_\sigma = 0{,}3$ the sinks are P_2, P_5 and P_7. The sources are P_1, P_4 and P_8. P_3, P_6, P_9 and P_{10} are saddles.



For $\gamma = 0$ and $c_\sigma = 0{,}6$ the sinks are P_2, P_3 and P_5. The sources are P_1, P_4 and P_6. P_7 and P_8 do not exist. The saddles are P_9 and P_{10}.

Phase space of the system (64).





For $\gamma = 1$ and $c_\sigma = 0{,}2$ the sinks are P_5 and P_7. The sources are P_4 and P_8. The saddles are P_1,P_2,P_3 and P_6. P_9 and P_{10} do not exist.

For $\gamma = 1$ and $c_\sigma = 0{,}6$ the sinks are P_3 and P_5. The sources are P_4 and P_6. P_1 and P_2 are saddles. The points P_7-P_{10} do not exist.

Let us consider the static case $\mathbf{e}_0(\cdot) = 0$, for a mixture of a perfect fluid and a scalar field. [In this case $\dot{u} \equiv d\ln N/dx \neq 0$ and the perfect fluid is forced to be non-tilted ($v = 0$).] We will consider barotropic equations of state $\hat{\mu} = \hat{\mu}(\hat{p})$. Since $\theta = \sigma_+ = 0$ in the static subcase, V depends only on the scalar field $\phi = \phi(x)$. Furthermore, the irreducible components of the scalar field energy-momentum tensor are given by
$\mu^\phi = \frac{1}{2}\mathbf{e}_1(\phi)^2 + V, p^\phi = -\frac{1}{6}\mathbf{e}_1(\phi)^2 - V, q^\phi = 0$ and $\pi_+^\phi = -\frac{1}{3}\mathbf{e}_1(\phi)^2$.
The equations for the variables $a, \dot{u}, \hat{p}, \phi, K, N$ are:

$$\mathbf{e}_1(a) = \frac{\hat{\mu} + 3\hat{p}}{2(c_a+1)} + \mathbf{e}_1(\phi)^2 - \frac{V}{c_a+1} + 2c_a\dot{u}^2 + 3a\dot{u} + K, \tag{65a}$$

$$\mathbf{e}_1(\dot{u}) = \frac{\hat{\mu} + 3\hat{p}}{2(c_a+1)} - \frac{V}{c_a+1} + 2a\dot{u} - \dot{u}^2, \tag{65b}$$

$$\mathbf{e}_1(\hat{p}) = -\dot{u}(\hat{\mu} + \hat{p}) \tag{65c}$$

$$\mathbf{e}_1(\mathbf{e}_1(\phi)) = -(\dot{u} - 2a)\,\mathbf{e}_1(\phi) + V_\phi, \tag{65d}$$

$$\mathbf{e}_1(K) = 2aK, \tag{65e}$$

where V_ϕ denotes differentiation with respect to ϕ. The system satisfies the restriction

$$a^2 = c_a\dot{u}^2 + 2a\dot{u} + \hat{p} + \frac{1}{2}\mathbf{e}_1(\phi)^2 - V + K. \tag{66}$$

We select an r-coordinate such that $\mathbf{e}_1(f) \equiv -ar\partial_r(f)$ (as in [Clarkson, Marklund, Betschart & Dunsby 2004]). This implies $K \propto r^{-2}$ and $\int {e_1}^1(x)^{-1}\mathrm{d}x = -\int a^{-1}\mathrm{d}\ln r$. As we will see later, for the dynamical systems investigation it is better to use the new time variable $\tau = \ln r$, which takes values over the whole real line. Here we shall study the two special cases:

(i) perfect fluid with linear equation of state

$$\hat{\mu} = \mu_0 + (\eta - 1)\hat{p}, \tag{67}$$

where the constants μ_0 and η satisfy $\mu_0 \geq 0, \eta \geq 1$ (previous work has assumed a comoving aether and a comoving fluid).

(ii) vacuum (stationary with a scalar field and harmonic potential $V(\phi) = \frac{m^2\phi^2}{2}$). In particular, for the stationary aether case, it is also of interest to choose a frame in which the aether is non-comoving.

Introducing the new dimensionless variables

$$x_1 = \frac{\mu_0}{a^2}, \quad x_2 = \frac{\dot{u}}{a}, \quad x_3 = \frac{\hat{p}}{a^2}, \quad x_4 = \frac{K}{a^2}, \tag{68}$$

subject to the constraint

$$1 = x_2^2 c_a + 2x_2 + x_4 + x_3. \tag{69}$$

$$\frac{dx_1}{d\tau} = \frac{x_1 x_3 (\eta - 2c_a)}{c_a + 1} + 2x_1 x_2^2 c_a + x_1 \left(\frac{x_1}{c_a + 1} + 2\right) + 2x_1 x_2, \tag{70a}$$

$$\frac{dx_2}{d\tau} = x_2 \left(\frac{x_1}{2c_a + 2} + x_3 \left(\frac{\eta + 2}{2c_a + 2} - 1\right) - 1\right) - \frac{x_1}{2c_a + 2} - \frac{x_3(\eta + 2)}{2c_a + 2} + x_2^3 c_a + 2x_2^2, \tag{70b}$$

$$\frac{dx_3}{d\tau} = x_3 \left(\frac{x_1}{c_a + 1} + 2\right) + x_3^2 \left(\frac{\eta + 2}{c_a + 1} - 2\right) + 2x_3 x_2^2 c_a + x_2(x_1 + x_3(\eta + 2)), \tag{70c}$$

defined on the phase space

$$\Psi = \left\{(x_1, x_2, x_3) : x_1 \geq 0, x_2^2 c_a + 2x_2 + x_3 \leq 1\right\}. \tag{71}$$

Equilibrium points of the system (70)

Label	x_1	x_2	x_3	Existence
P_1	0	0	0	always
P_2	$-2-\eta$	0	1	$\notin \Psi$
P_3	0	$-\frac{2}{\eta}$	$\frac{4(c_a+1)}{\eta^2}$	$c_a \leq \Delta_1$
P_4	0	$\frac{\eta+2}{-4c_a+\eta-2}$	$-\frac{(c_a+1)((\eta-2)(\eta+6)-16c_a)}{(-4c_a+\eta-2)^2}$	$1 \leq \eta < 2, \Delta_2 \leq c_a < \frac{\eta-2}{4}$ or $\eta > 2, \frac{\eta-2}{4} < c_a \leq \Delta_2$
P_5	$-\frac{\eta(c_a+1)(4c_a+3)}{(2c_a+1)^2}$	$\frac{1}{-2c_a-1}$	$\frac{(c_a+1)(4c_a+3)}{(2c_a+1)^2}$	$\eta \geq 1, -1 \leq c_a \leq -\frac{3}{4}$
$P_{6,7}$	0	$\frac{1}{1\pm\sqrt{1+c_a}}$	0	always

$\Delta_1 = \frac{1}{8}\left(\eta^2+4\eta-4\right)$ and $\Delta_2 = \frac{1}{16}\left(\eta^2+4\eta-12\right)$.

The equilibrium points of the system (70) and their stability

1. The equilibrium point P_1 is always a saddle. It satisfies $K = a^2$ asymptotically, which implies ${e_1}^1 \sim e^{-\tau} = \frac{1}{r}, {e_2}^2 \sim e^{-\tau} = \frac{1}{r}$. Since $\dot{u} \ll a$ as $r \to \infty$, it follows that $N \ll r^{-1}$.
2. Although the equilibrium point P_2 can be an attractor for $\eta \geq 1, c_a \leq -\eta - 3$ or $\eta \geq 1, -\eta - 3 < c_a < -1$, since it can never belong to the phase space (denoted $\notin \Psi_2$ in table), we do not discuss it further.
3. The equilibrium point P_3 is always a saddle.
4. The equilibrium point P_4 is a source for $\eta > 2$, $\frac{\eta-2}{4} < c_a < \frac{1}{16}(\eta - 2)(\eta + 6)$. Otherwise it is a saddle.
5. The equilibrium point P_5 is a saddle for $\eta \geq 1, -1 \leq c_a < -\frac{3}{4}$. It is non-hyperbolic for $c_a = -\frac{3}{4}$ [but it behaves as a saddle].
6. P_6 is a source for $\eta \geq 1, -1 < c_a < 0$, or $\eta \geq 1, c_a > 0$.
7. P_7 is a sink for $1 \leq \eta < 2$, $\frac{1}{16}(\eta - 2)(\eta + 6) < c_a < 0$. It is a source for $1 \leq \eta \leq 2, c_a > 0$ or $\eta > 2, c_a > \frac{1}{16}(\eta - 2)(\eta + 6)$. It is a saddle otherwise.

Model (ii): Static vacuum aether with a scalar field with harmonic potential.

Let us investigate a static vacuum aether with a scalar field with harmonic potential $V(\phi) = \frac{m^2\phi^2}{2}$. Introducing the new dimensionless variables

$$x_2 = \frac{\dot{u}}{a}, \quad x_4 = \frac{K}{a^2}, \quad x_5 = \frac{\sqrt{2}m}{a}, \quad x_6 = \frac{\mathbf{e}_1(\phi)}{a}, \quad x_7 = \frac{m\phi}{\sqrt{2}a}, \tag{72}$$

subject to the constraint $1 = c_a {x_2}^2 + 2x_2 + x_4 + \frac{1}{2}x_6^2 - x_7^2$.

$$\frac{dx_2}{d\tau} = x_2^3 c_a - \frac{(x_2 - 1)x_7^2}{c_a + 1} + \frac{1}{2}x_2\left(4x_2 + 2x_7^2 + x_6^2 - 2\right), \tag{73a}$$

$$\frac{dx_5}{d\tau} = \frac{1}{2}x_5\left(2c_a\left(\frac{x_7^2}{c_a+1} + x_2^2\right) + 2x_2 + x_6^2 + 2\right), \tag{73b}$$

$$\frac{dx_6}{d\tau} = x_6 c_a\left(\frac{x_7^2}{c_a+1} + x_2^2\right) + \frac{1}{2}\left(x_6\left(4x_2 + x_6^2 - 2\right) - 2x_5 x_7\right), \tag{73c}$$

$$\frac{dx_7}{d\tau} = x_7 c_a\left(\frac{x_7^2}{c_a+1} + x_2^2\right) - \frac{x_5 x_6}{2} + x_7\left(x_2 + \frac{x_6^2}{2} + 1\right), \tag{73d}$$

defined in the phase space

$$\Psi = \left\{\left(x_2, x_5, x_6, x_7\right) : c_a {x_2}^2 + 2x_2 + x_4 + \frac{1}{2}x_6^2 - x_7^2 \leq 1\right\}. \tag{74}$$

Equilibrium points of the system (73).

Label	x_2	x_5	x_6	x_7	Existence
Q_1	0	0	0	0	always
$Q_{2,3}$	$x_2^\star$	0	$\pm\sqrt{2}\sqrt{1-2x_2^\star-c_a{x_2^\star}^2}$	0	$1-2x_2^\star-c_a{x_2^\star}^2\geq 0$
$Q_{4,5}$	$-\frac{1}{2c_a+1}$	0	0	$\pm\frac{\sqrt{(-c_a-1)(4c_a+3)}}{2c_a+1}$	$(c_a+1)(4c_a+3)\leq 0$
$Q_{6,7}$	$-\frac{1}{2c_a+1}$	0	$\pm\frac{\sqrt{2}\sqrt{(c_a+1)(4c_a+3)}}{2c_a+1}$	0	$(c_a+1)(4c_a+3)\geq 0$
$Q_{8,9}$	$\frac{1}{1\pm\sqrt{c_a+1}}$	0	0	0	always

$x_2^\star$ is a parameter and hence the curves $Q_{2,3}$ represent lines of equilibrium points ($x_2^\star=0, x_2^\star=2$ are special points on these curves).

Stability

1. Q_1 is always a saddle.
2. The line of equilibrium points $Q_{2,3}$ is normally hyperbolic and is stable when $x_2^\star > 2$.
3. The equilibrium points $Q_{4,5}$ are non-hyperbolic. They have a 3D stable manifold and a 1D center manifold for $-1 \leq c_a < -\frac{3}{4}$ and a 1D stable manifold and a 3D center manifold for $c_a = -\frac{3}{4}$.
4. $Q_{6,7}$ are non-hyperbolic. They have a 3D stable manifold and a 1D center manifold for $-\frac{3}{4} < c_a < -\frac{1}{2}$. They have a 3D unstable manifold and a 1D center manifold for $c_a < -1$ or $c_a > -\frac{1}{2}$ [the non zero eigenvalues are always of the same sign].
5. Q_8 is non-hyperbolic. It has a 3D unstable manifold and a 1D center manifold for $-1 < c_a < 0$ or $c_a > 0$. Otherwise, its center manifold has dimension greater than 1.
6. Q_9 is non-hyperbolic. It has a 3D stable manifold and a 1D center manifold for $-\frac{3}{4} < c_a < 0$. It has a 3D unstable manifold and a 1D center manifold for $c_a > 0$. Finally, Q_9 has a 2D unstable manifold, a 1D center manifold and a 1D stable manifold for $-1 < c_a < -\frac{3}{4}$.

Conclusions

- We have studied spherically symmetric Einstein-aether models with tilting perfect fluid matter, which are also solutions of the IR limit of Horava gravity [Jacobson 2013].
- We used the 1+3 frame formalism to write down the evolution equations for non-comoving perfect fluid spherically symmetric models and showed they form a well-posed system of first order PDEs in two variables.
- We adopted the so-called comoving aether gauge (which implies a preferred foliation, the only remaining freedom is the coordinate time and space reparameterization freedom). We also introduced (β-) normalized variables. In particular, we considered the special subset $\dot{U} = v = 0$ (where we also assumed $c_\theta = 0$ and $c_\sigma \neq 0$) and derived the final reduced phase space equations in normalized variables.
- We have studied inhomogeneous cosmologies in Einstein-aether theories of gravity.
- We first examined the conditions for the existence of McVittie-like solutions in the context of Einstein-aether theory. We found that they only exist for the choice of parameters $c_a = 0, \gamma = 0$, and for an aligned aether ($v = 0$). Since $\gamma = 0$, the matter fluid corresponds to a cosmological constant (and θ is always a constant). Irrespective of the sign of the initial expansion, the physical variables tend to zero as $t \to +\infty$.

- We then considered dust models. We investigated a special dust model with $\dot{U} = 0$ and $v = 0$ in normalized variables (assuming $c_\sigma \neq 0$) and derived a reduced (closed) evolution system. The FLRW models in this special dust model correspond to an equilibrium point.
- We paid particular attention to the sinks for different values of the parameter c_σ. In all cases $\Omega \to 0$ to the future. For all solutions with small $c_\sigma < 3/8$, $\mathscr{D} \to 1$ ($\mathscr{Q} \to 1$) and the shear goes to zero at late times.
- We then considered the spatially homogeneous Kantowski-Sachs models using appropriate normalized variables, and obtained the general evolution equations. We then considered a special case with $3c_\theta \equiv c_1 + 3c_2 + c_3 = 0$ and analysed the qualitative behaviour.
- In the case $c_\sigma < \frac{1}{2}$ (i.e., $c > 0$), when $\gamma < \frac{2}{3}$, there is the unique shear-free, zero curvature (FLRW) inflationary future attractor (P_2), and for $\frac{3}{8} < c_\sigma < \frac{1}{2}$ (i.e., $0 < c < \frac{1}{2}$) and $0 \leq \gamma < 2$ all of the sources and sinks (respectively, P_4 & P_8 and P_5 & P_7) have maximal shearing and all except one sink (P_7) have zero curvature. For $c_\sigma < \frac{3}{8}$ (i.e., $c > \frac{1}{2}$), the points P_7 & P_8 do not exist, and the sources and sinks with maximal shearing are P_4 and P_5, respectively.

- Finally, we considered static models for a mixture of a (necessarily non-tilted with $v = 0$) perfect fluid with a barotropic equations of state and a scalar field (with a self-interaction potential V that depends only on the scalar field).
- In particular, we studied the special cases of a tilted perfect fluid and no scalar field (previous work had assumed a comoving aether and a comoving fluid) and a stationary vacuum with a scalar field (with a harmonic potential).
- The equilibrium points in the resulting dynamical systems in these two cases were determined and their stability was investigated. The physical interpretation of this analysis will be discussed in a forthcoming paper.

Future work

- Investigate the general Kantowski-Sachs models and the static models more comprehensively.
- Determine the structure of stationary rotating solutions; rapidly rotating black holes, unlike the non-rotating ones, might turn out to be very different from the Kerr metrics of GR.

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