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Cosmology in Delta Gravity: A Classical Analysis and Phenomenology.

by
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What is Delta Gravity?

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2. We present a gauge theories, $\tilde{\delta}$ gauge theories (DGT), which main properties are:
   - New kind of fields are created, $\tilde{\phi}_I$, from the originals ϕ_I.
   - The classical equations of motion of ϕ_I are satisfied in the full quantum theory.
   - The model lives at one loop.
   - The action is obtained through the extension of the original gauge symmetry of the model, introducing an extra symmetry that we call $\tilde{\delta}$ symmetry, since it is formally obtained as the variation of the original symmetry.

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   - The model lives at one loop.
   - The action is obtained through the extension of the original gauge symmetry of the model, introducing an extra symmetry that we call $\tilde{\delta}$ symmetry, since it is formally obtained as the variation of the original symmetry.

When we apply this prescription to GR, we obtain $\tilde{\delta}$ gravity. To Quantum Delta-Gravity, see: **Jorge Alfaro, Ricardo Ávila and Pablo González 2011 Class. Quantum Grav. 28 215020**.

Overview:

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$\tilde{\delta}$ Theories.

$\tilde{\delta}$ Variation:

These theories consist in the application of a variation that we will call $\tilde{\delta}$. when we applied it on a field, it will give new elements that we define as $\tilde{\delta}$ field. As a variation, it will have all properties of an usual variation.

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$$\begin{aligned}
\tilde{\delta}(AB) &= \tilde{\delta}(A)B + A\tilde{\delta}(B) \\
\tilde{\delta}\delta A &= \delta\tilde{\delta} A \\
\tilde{\delta}(\Phi_{,\mu}) &= (\tilde{\delta}\Phi)_{,\mu}
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We use the convention that a tilde tensor is equal to the $\tilde{\delta}$ transformation of the original tensor associated to it when all its indexes are covariant:

$$\tilde{S}_{\mu\nu\alpha\ldots} \equiv \tilde{\delta}\left(S_{\mu\nu\alpha\ldots}\right)$$

and we raise and lower indexes using the metric $g_{\mu\nu}$.

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$$\tilde{S}_{\mu\nu\alpha\ldots} \equiv \tilde{\delta}\left(S_{\mu\nu\alpha\ldots}\right) \quad (2)$$

and we raise and lower indexes using the metric $g_{\mu\nu}$. Therefore:

$$\begin{aligned} \tilde{\delta}\left(S^{\mu}{}_{\nu\alpha\ldots}\right) &= \tilde{\delta}(g^{\mu\rho}S_{\rho\nu\alpha\ldots}) \\ &= \tilde{\delta}(g^{\mu\rho})S_{\rho\nu\alpha\ldots} + g^{\mu\rho}\tilde{\delta}\left(S_{\rho\nu\alpha\ldots}\right) \\ &= -\tilde{g}^{\mu\rho}S_{\rho\nu\alpha\ldots} + \tilde{S}^{\mu}{}_{\nu\alpha\ldots} \end{aligned} \quad (3)$$

$\tilde{\delta}$ Transformation:

In $\tilde{\delta}$ Theories, the transformations are:

$$\begin{aligned} \bar{\delta}\Phi_i &= \Lambda_i^j(\Phi)\epsilon_j \\ \bar{\delta}\tilde{\Phi}_i &= \tilde{\Lambda}_i^j(\Phi)\epsilon_j + \Lambda_i^j(\Phi)\tilde{\epsilon}_j \end{aligned}$$

Where we used that $\tilde{\delta}\bar{\delta}\Phi_i = \bar{\delta}\tilde{\delta}\Phi_i = \bar{\delta}\tilde{\Phi}_i$ and $\tilde{\epsilon}_j = \tilde{\delta}\epsilon_j$.

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Where we used that $\tilde{\delta}\bar{\delta}\Phi_i = \bar{\delta}\tilde{\delta}\Phi_i = \bar{\delta}\tilde{\Phi}_i$ and $\tilde{\epsilon}_j = \tilde{\delta}\epsilon_j$. Now, if we consider general coordinate transformations in its infinitesimal form:

$$\begin{aligned} x'^{\mu} &= x^{\mu} - \xi_0^{\mu}(x) \\ \bar{\delta}x^{\mu} &= -\xi_0^{\mu}(x) \end{aligned}$$

and defining $\xi_1^{\mu}(x) \equiv \tilde{\delta}\xi_0^{\mu}(x)$, we can see that, for scalar fields:

$$\bar{\delta}\phi = \xi_0^{\mu}\phi_{,\mu} \tag{4}$$

$$\bar{\delta}\tilde{\phi} = \xi_1^{\mu}\phi_{,\mu} + \xi_0^{\mu}\tilde{\phi}_{,\mu} \tag{5}$$

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$$\bar{\delta}V_{\mu} = \xi_0^{\beta}V_{\mu,\beta} + \xi_{0,\mu}^{\alpha}V_{\alpha} \quad (6)$$
$$\bar{\delta}\tilde{V}_{\mu} = \xi_1^{\beta}V_{\mu,\beta} + \xi_{1,\mu}^{\alpha}V_{\alpha} + \xi_0^{\beta}\tilde{V}_{\mu,\beta} + \xi_{0,\mu}^{\alpha}\tilde{V}_{\alpha} \quad (7)$$

$\tilde{\delta}$ Transformation:

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$$\begin{aligned} \bar{\delta}\Phi_i &= \Lambda_i^j(\Phi)\epsilon_j && (8) \\ \bar{\delta}\tilde{\Phi}_i &= \tilde{\Lambda}_i^j(\Phi)\epsilon_j + \Lambda_i^j(\Phi)\tilde{\epsilon}_j && (9) \end{aligned}$$

Where we used that $\tilde{\delta}\bar{\delta}\Phi_i = \bar{\delta}\tilde{\delta}\Phi_i = \bar{\delta}\tilde{\Phi}_i$ and $\tilde{\epsilon}_j = \tilde{\delta}\epsilon_j$. Now, if we consider general coordinate transformations in its infinitesimal form:

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and defining $\xi_1^{\mu}(x) \equiv \tilde{\delta}\xi_0^{\mu}(x)$, we can see that, for rank two covariant fields:

$$\begin{aligned} \bar{\delta}M_{\mu\nu} &= \xi_0^{\rho}M_{\mu\nu,\rho} + \xi_{0,\nu}^{\beta}M_{\mu\beta} + \xi_{0,\mu}^{\beta}M_{\nu\beta} && (11) \\ \bar{\delta}\tilde{M}_{\mu\nu} &= \xi_1^{\rho}M_{\mu\nu,\rho} + \xi_{1,\nu}^{\beta}M_{\mu\beta} + \xi_{1,\mu}^{\beta}M_{\nu\beta} + \xi_0^{\rho}\tilde{M}_{\mu\nu,\rho} + \xi_{0,\nu}^{\beta}\tilde{M}_{\mu\beta} + \xi_{0,\mu}^{\beta}\tilde{M}_{\nu\beta} && (12) \end{aligned}$$

$\tilde{\delta}$ Gravity Action.

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The Invariant action under our transformations is:

$$S[\phi, \tilde{\phi}] = S_0[\phi] + \kappa_2 \int \frac{\delta S_0}{\delta \phi_I(x)}[\phi]\tilde{\phi}_I(x)$$

where κ_2 is an arbitrary constant, the index I refers to any kind of indices and $\tilde{\phi}_I(x) = \tilde{\delta}\phi_I(x)$. This new action shows the standard structure which is used to define any modified element or function for $\tilde{\delta}$ type models. In particular, let us consider the Einstein-Hilbert Action.

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$$S_0 = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} + L_M\right) \tag{14}$$

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} + L_M - \frac{\kappa_2}{2\kappa}\left(G^{\mu\nu} - \kappa T^{\mu\nu}\right)\tilde{g}_{\mu\nu} + \kappa_2 \tilde{L}_M\right) \tag{15}$$

with $\kappa = \frac{8\pi G}{c^2}$ and:

$$\tilde{L}_M = \tilde{\phi}_I \frac{\delta L_M}{\delta \phi_I} + (\partial_\mu \tilde{\phi}_I)\frac{\delta L_M}{\delta(\partial_\mu \phi_I)} \tag{16}$$

Equations of Motion:

If we vary $\tilde{g}_{\mu\nu}$ in (15), we obtain the equation of motion:

$$G^{\mu\nu} = \kappa T^{\mu\nu}. \tag{17}$$

The Einstein's equation are preserved!!!

Equations of Motion:

Now, if we vary $g_{\mu\nu}$ in (15), we obtain a new equation. This equation give us $\tilde{g}_{\mu\nu}$. It is simplified to:

$$F^{(\mu\nu)(\alpha\beta)\rho\lambda} D_\rho D_\lambda \tilde{g}_{\alpha\beta} + \frac{1}{2}\left(g^{\mu\nu} R^{\alpha\beta} \tilde{g}_{\alpha\beta} - \tilde{g}^{\mu\nu} R\right) = \kappa \tilde{T}^{\mu\nu}$$

With $\tilde{T}_{\mu\nu} = \tilde{\delta} T_{\mu\nu}$, so $\tilde{T}^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \tilde{\delta} T_{\alpha\beta}$. And:

$$\begin{aligned} F^{(\mu\nu)(\alpha\beta)\rho\lambda} &= P^{((\rho\mu)(\alpha\beta))} g^{\nu\lambda} + P^{((\rho\nu)(\alpha\beta))} g^{\mu\lambda} - P^{((\mu\nu)(\alpha\beta))} g^{\rho\lambda} \\ &\quad - P^{((\rho\lambda)(\alpha\beta))} g^{\mu\nu} \\ P^{((\alpha\beta)(\mu\nu))} &= \frac{1}{4}\left(g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - g^{\alpha\beta} g^{\mu\nu}\right) \end{aligned}$$

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With $\tilde{T}_{\mu\nu} = \tilde{\delta}T_{\mu\nu}$, so $\tilde{T}^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\tilde{\delta}T_{\alpha\beta}$. And:

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Besides, the symmetries give us two conservation rules:

$$D_\nu T^{\mu\nu} = 0 \tag{20}$$

$$D_\nu \tilde{T}^{\mu\nu} = \frac{1}{2}T^{\alpha\beta}D^\mu \tilde{g}_{\alpha\beta} - \frac{1}{2}T^{\mu\beta}D_\beta \tilde{g}^\alpha_\alpha + D_\beta(\tilde{g}^\beta_\alpha T^{\alpha\mu}) \tag{21}$$

$\tilde{\delta}$ Free Particle Action.

Massive Free Particle Action:

In the standard case, the test particle action is:

$$S_0[\dot{x}, g] = -m \int dt \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

But, in $\tilde{\delta}$ gravity, we will have a new test particle action. To obtain this action, we need to use (13):

$$S[\dot{x}, y, g, \tilde{g}] = m \int dt \frac{\bar{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{\kappa_2}{2}(2g_{\mu\nu}\dot{y}^\mu \dot{x}^\nu + g_{\mu\nu,\rho}y^\rho \dot{x}^\mu \dot{x}^\nu)}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}}$$

where $\bar{g}_{\mu\nu} = g_{\mu\nu} + \frac{\kappa_2}{2}\tilde{g}_{\mu\nu}$ and $y^\mu = \tilde{\delta}x^\mu$. This action is invariant under extended reparametrization:

$$\begin{aligned} \delta x^\mu &= \dot{x}^\mu \epsilon \\ \delta y^\mu &= \dot{y}^\mu \epsilon + \dot{x}^\mu \tilde{\epsilon} \end{aligned}$$

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where $\bar{g}_{\mu\nu} = g_{\mu\nu} + \frac{\kappa_2}{2}\tilde{g}_{\mu\nu}$ and $y^\mu = \tilde{\delta}x^\mu$. This action is invariant under extended reparametrization:

$$\begin{aligned} \delta x^\mu &= \dot{x}^\mu \epsilon \\ \delta y^\mu &= \dot{y}^\mu \epsilon + \dot{x}^\mu \tilde{\epsilon} \quad \textbf{This symmetry is eliminated.} \end{aligned} \tag{23}$$

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$$S[\dot{x}, g, \tilde{g}] = m \int dt \left(\frac{\bar{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} \right) \tag{24}$$

where $\bar{g}_{\mu\nu} = g_{\mu\nu} + \frac{\kappa_2}{2}\tilde{g}_{\mu\nu}$. If we vary (24) with respect to x^μ, we obtain the equation of motion for a massive test particle:

$$\hat{g}_{\mu\nu}\ddot{x}^\nu + \hat{\Gamma}_{\mu\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \frac{\kappa_2}{4}\tilde{K}_{,\mu} \tag{25}$$

with $\hat{\Gamma}_{\mu\alpha\beta} = \frac{1}{2}(\hat{g}_{\mu\alpha,\beta} + \hat{g}_{\beta\mu,\alpha} - \hat{g}_{\alpha\beta,\mu})$, $\hat{g}_{\alpha\beta} = \left(1 + \frac{\kappa_2}{2}\tilde{K}\right) g_{\alpha\beta} + \kappa_2 \tilde{g}_{\alpha\beta}$ and $\tilde{K} = \tilde{g}_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$. Besides, we fix $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$, after choosing t equal to the proper time.

Massless Free Particle Action:

Unfortunately, (24) is useless for massless particles, because it is null when $m = 0$. To solve this problem, it is usual to start from the action:

$$S_0[\dot{x}, g, v] = \frac{1}{2}\int dt \left(vm^2 - v^{-1}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)$$

where v is a Lagrange multiplier.

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where v is a Lagrange multiplier. To find the coupling of a test particle to a gravitational background field, we follow the prescription to construct $\tilde{\delta}$ models. So, if we evaluate (26) in (13), we obtain:

$$\begin{aligned} S[\dot{x}, g, \tilde{g}, v, \tilde{v}] &= \frac{1}{2}\int dt[vm^2 - v^{-1}\left(g_{\mu\nu} + \kappa_2\tilde{g}_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu \\ &\quad +\kappa_2\tilde{v}\left(m^2 + v^{-2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)] \end{aligned}$$

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Two Lagrange multiplier are unnecessary, so we will eliminate one of them. The equation of motion for $\tilde{v}$ is:

$$\tilde{v} = \frac{m^2 + v^{-2} \left(g_{\mu\nu} + \kappa_2 \tilde{g}_{\mu\nu} \right) \dot{x}^\mu \dot{x}^\nu}{2\kappa_2 v^{-3} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}$$

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where v is a Lagrange multiplier. To find the coupling of a test particle to a gravitational background field, we follow the prescription to construct $\tilde{\delta}$ models. Finally, our modified free particle action is:

$$S[\dot{x}, g, \tilde{g}, v] = \tag{27}$$
$$\int dt\left(m^2 v - \frac{\left(g_{\mu\nu} + \kappa_2\tilde{g}_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu}{4v} + \frac{m^2 v^3}{4g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}\left(m^2 + \kappa_2 v^{-2}\tilde{g}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)\right)$$

When $m \neq 0$, we can reduced (27) to (24) using the equation of motion of v. That is:

$$v = -\frac{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}{m} \tag{28}$$

Geodesic:

Evaluating $m = 0$ in (26) and (27), they respectively are:

$$S_0^{(m=0)}[\dot{x}, g, v] = -\frac{1}{2}\int dt v^{-1} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

$$S^{(m=0)}[\dot{x}, g, \tilde{g}, v] = -\frac{1}{4}\int dt v^{-1} \mathbf{g}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

with $\mathbf{g}_{\mu\nu} = g_{\mu\nu} + \kappa_2\tilde{g}_{\mu\nu}$. Therefore, in the usual case we have $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$, but in our model the null-geodesic is $\mathbf{g}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$.

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with $\mathbf{g}_{\mu\nu} = g_{\mu\nu} + \kappa_2 \tilde{g}_{\mu\nu}$. Therefore, in the usual case we have $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$, but in our model the null-geodesic is $\mathbf{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$.

Summary:

To massive particle, $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ is conserved, so we can choose t equal to the proper time. Therefore, we measure proper time using the metric $g_{\mu\nu}$. On the other side, if we consider the motion of light rays along infinitesimally near trajectories, the three-dimensional metric is:

$$dl^2 = \left(\frac{g_{00}}{\mathbf{g}_{00}}\left(\mathbf{g}_{ij} - \frac{\mathbf{g}_{i0}\mathbf{g}_{j0}}{\mathbf{g}_{00}}\right)\right) dx^i dx^j \quad (31)$$

Extended Harmonic gauge.

Extended harmonic gauge:

We know that the Einstein's equations do not fix all degrees of freedom of $g_{\mu\nu}$. This means that, if $g_{\mu\nu}$ is solution, then exist other solution $g'_{\mu\nu}$ given by a general coordinate transformation $x \to x'$. We can eliminate these degrees of freedom by adopting some particular coordinate system, fixing the gauge. One particularly convenient gauge is given by the harmonic coordinate conditions.

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$$\Gamma^{\mu} \equiv g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta} = 0$$

It is always possible to choose an harmonic coordinate system. In the same form, we need to fix the gauge for $\tilde{g}_{\mu\nu}$.

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It is always possible to choose an harmonic coordinate system. In the same form, we need to fix the gauge for $\tilde{g}_{\mu\nu}$. It is natural to choose a gauge given by:

$$\tilde{\delta}\left(\Gamma^{\mu}\right) \equiv g^{\alpha\beta}\tilde{\delta}\left(\Gamma^{\mu}_{\alpha\beta}\right) - \tilde{g}^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta} = 0 \tag{33}$$

Where $\tilde{\delta}\left(\Gamma^{\mu}_{\alpha\beta}\right) = \frac{1}{2}g^{\mu\lambda}\left(D_{\beta}\tilde{g}_{\lambda\alpha} + D_{\alpha}\tilde{g}_{\beta\lambda} - D_{\lambda}\tilde{g}_{\alpha\beta}\right)$. So, when we will refer to Extended harmonic gauge, we will use (32) and (33).

Schwarzschild Case.

Vacuum Solution:

In this case, we have that:

$$
\begin{aligned}
g_{\mu\nu}dx^\mu dx^\nu &= -A(r)c^2dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \\
\tilde{g}_{\mu\nu}dx^\mu dx^\nu &= -\tilde{A}(r)c^2dt^2 + \tilde{B}(r)dr^2 + \tilde{F}(r)r^2(d\theta^2 + \sin^2(\theta)d\phi^2)
\end{aligned}
$$

To simplify the equations, we will solve them outside matter, this means the region where $\tilde{T}_{\mu\nu} = T_{\mu\nu} = 0$. To find $\tilde{A}(r)$ and $\tilde{F}(r)$, we need an additional equation, fixing the gauge for $\tilde{g}_{\mu\nu}$. The extended harmonic gauge say us that:

$$
\begin{aligned}
r^2(r-2\mu)\tilde{A}''(r) &+ 4r(r-2\mu)\tilde{A}'(r) - 4\mu\tilde{A}(r) \\
&+ r(r-2\mu)(r-\mu)\tilde{F}''(r) + 4(r-\mu)^2\tilde{F}'(r) = 0
\end{aligned}
$$

Vacuum Solution:

In this case, we have that:

$$
\begin{aligned}
g_{\mu\nu}dx^\mu dx^\nu &= -A(r)c^2dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \\
\tilde{g}_{\mu\nu}dx^\mu dx^\nu &= -\tilde{A}(r)c^2dt^2 + \tilde{B}(r)dr^2 + \tilde{F}(r)r^2(d\theta^2 + \sin^2(\theta)d\phi^2)
\end{aligned}
$$

So, the complete solution is:

$$
\begin{aligned}
A(r) &= 1 - \frac{2\mu}{r} \\
B(r) &= \left(1 - \frac{2\mu}{r}\right)^{-1} \\
\tilde{A}(r) &= -\frac{2a_0\mu(r-\mu)}{r^2} - \frac{a_1\mu\left(2\mu + (r-\mu)\ln\left(1-\frac{2\mu}{r}\right)\right)}{r^2} \\
\tilde{B}(r) &= \frac{2a_0\mu(r-\mu)}{(r-2\mu)^2} - \frac{a_1\left(2\mu(r-2\mu) + (r^2 - 3\mu r + \mu^2)\ln\left(1-\frac{2\mu}{r}\right)\right)}{(r-2\mu)^2} \\
\tilde{F}(r) &= \frac{2a_0\mu}{r} - \frac{a_1\left(2\mu + (r-\mu)\ln\left(1-\frac{2\mu}{r}\right)\right)}{r}
\end{aligned}
$$

Vacuum Solution:

In this case, we have that:

$$
\begin{aligned}
g_{\mu\nu}dx^{\mu}dx^{\nu} &= -A(r)c^2dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \\
\tilde{g}_{\mu\nu}dx^{\mu}dx^{\nu} &= -\tilde{A}(r)c^2dt^2 + \tilde{B}(r)dr^2 + \tilde{F}(r)r^2(d\theta^2 + \sin^2(\theta)d\phi^2)
\end{aligned}
$$

We must remember that this solution corresponds to the region without matter, so $r > R$, and generally the Newtonian approximation, $R \gg 2\mu$, can be used. So, we must consider the leading order in $\frac{\mu}{r}$.

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\end{aligned}
$$

We must remember that this solution corresponds to the region without matter, so $r > R$, and generally the Newtonian approximation, $R \gg 2\mu$, can be used. So, we must consider the leading order in $\frac{\mu}{r}$. That is:

$$
\begin{aligned}
A(r) &= 1 - \frac{2\mu}{r} \\
B(r) &= 1 + \frac{2\mu}{r} + O\left(\left(\frac{\mu}{r}\right)^2\right) \\
\tilde{A}(r) &= -\frac{2a_0\mu}{r} + O\left(\left(\frac{\mu}{r}\right)^2\right) \\
\tilde{B}(r) &= \frac{2a_0\mu}{r} + O\left(\left(\frac{\mu}{r}\right)^2\right) \\
\tilde{F}(r) &= \frac{2a_0\mu}{r} + O\left(\left(\frac{\mu}{r}\right)^2\right)
\end{aligned}
$$

Gravitational Lensing:

To describe this phenomenon, we need the null geodesic. To solve these equations, we will consider a coordinate system where $\theta = \frac{\pi}{2}$[1].

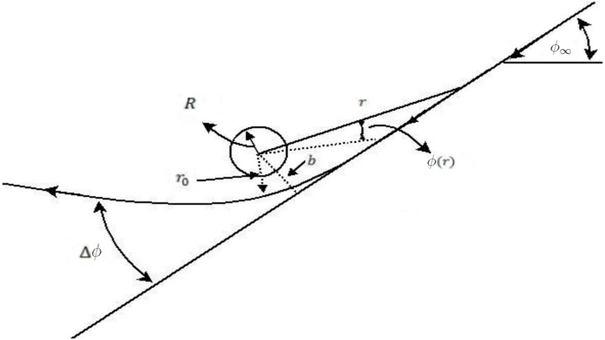

Figure: Trajectory by gravitational lensing. R is the radius of the star, r_0 is the minimal distance to the star, b is the impact parameter, ϕ_∞ is the incident direction and $\Delta\phi$ is the deflection of light.

[1] For instance, see S. Weinberg. Massachusetts Institute of Technology (1972), Chapter 8.

Gravitational Lensing:

To describe this phenomenon, we need the null geodesic. To solve these equations, we will consider a coordinate system where $\theta = \frac{\pi}{2}$. So, the geodesic equations are reduced to:

$$\begin{aligned}
\frac{dt}{du} &= \frac{1}{A(r) + \kappa_2 \tilde{A}(r)} \\
\frac{dr}{du} &= -\frac{1}{r}\sqrt{\frac{r^2(1+\kappa_2\tilde{F}(r)) - J^2(A(r)+\kappa_2\tilde{A}(r))}{(A(r)+\kappa_2\tilde{A}(r))(B(r)+\kappa_2\tilde{B}(r))(1+\kappa_2\tilde{F}(r))}} \\
\frac{d\phi}{du} &= \frac{J}{r^2(1+\kappa_2\tilde{F}(r))}
\end{aligned}$$

Where u is the trajectory parameter such that $x^\mu = x^\mu(u)$. We have fixed t such that $t \to u$ for $r \to \infty$ and J is a constant of motion related to the angular momentum:

$$J = r_0\sqrt{\frac{1+\kappa_2\tilde{F}(r_0)}{A(r_0)+\kappa_2\tilde{A}(r_0)}} \quad (34)$$

Gravitational Lensing:

So, the trajectory is given by:

$$\phi(r) - \phi_\infty = \int_r^\infty \sqrt{\frac{(A(r) + \kappa_2 \tilde{A}(r))(B(r) + \kappa_2 \tilde{B}(r))}{(1 + \kappa_2 \tilde{F}(r))(A(r_0) + \kappa_2 \tilde{A}(r_0))\left(\frac{r^2(1+\kappa_2 \tilde{F}(r))}{1+\kappa_2 \tilde{F}(r_0)} - \frac{r_0^2(A(r)+\kappa_2 \tilde{A}(r))}{A(r_0)+\kappa_2 \tilde{A}(r_0)}\right)}} \times \left(\frac{r_0}{r}\right) dr$$

We must use the approximation $r \geq r_0 \gg \mu$ for the solar system.

Gravitational Lensing:

So, the trajectory is given by:

$$\phi(r) - \phi_\infty =$$
$$\int_r^\infty \sqrt{\frac{(A(r) + \kappa_2 \tilde{A}(r))(B(r) + \kappa_2 \tilde{B}(r))}{(1 + \kappa_2 \tilde{F}(r))(A(r_0) + \kappa_2 \tilde{A}(r_0)) \left(\frac{r^2(1+\kappa_2 \tilde{F}(r))}{1+\kappa_2 \tilde{F}(r_0)} - \frac{r_0^2(A(r)+\kappa_2 \tilde{A}(r))}{A(r_0)+\kappa_2 \tilde{A}(r_0)} \right)}}$$
$$\times \left(\frac{r_0}{r}\right) dr$$

We must use the approximation $r \geq r_0 \gg \mu$ for the solar system. That is:

$$\phi(r) - \phi_\infty \simeq \int_r^\infty \frac{dr}{r} \left(1 + \frac{\mu}{r} + \frac{\mu(1 + 2\kappa_2 a_0)r}{r_0(r + r_0)}\right) \left(\left(\frac{r}{r_0}\right)^2 - 1\right)^{-\frac{1}{2}}$$

Gravitational Lensing:

So, the trajectory is given by:

$$\phi(r) - \phi_\infty =$$
$$\int_r^\infty \sqrt{\frac{(A(r)+\kappa_2\tilde{A}(r))(B(r)+\kappa_2\tilde{B}(r))}{(1+\kappa_2\tilde{F}(r))(A(r_0)+\kappa_2\tilde{A}(r_0))\left(\frac{r^2(1+\kappa_2\tilde{F}(r))}{1+\kappa_2\tilde{F}(r_0)} - \frac{r_0^2(A(r)+\kappa_2\tilde{A}(r))}{A(r_0)+\kappa_2\tilde{A}(r_0)}\right)}}$$
$$\times\left(\frac{r_0}{r}\right)dr$$

We must use the approximation $r \geq r_0 \gg \mu$ for the solar system. That is:

$$\phi(r) - \phi_\infty \simeq \int_r^\infty \frac{dr}{r}\left(1+\frac{\mu}{r}+\frac{\mu(1+2\kappa_2 a_0)r}{r_0(r+r_0)}\right)\left(\left(\frac{r}{r_0}\right)^2 - 1\right)^{-\frac{1}{2}}$$

The deflection of light is given by:

$$\Delta\phi = 2|\phi(r_0) - \phi_\infty| - \pi$$

Gravitational Lensing:

So:

$$\begin{aligned}\Delta\phi &= 2|\phi(r_0) - \phi_\infty| - \pi \\ &\simeq 2\left|\int_{r_0}^{\infty} \frac{dr}{r}\left(1 + \frac{\mu(2\kappa_2 a_0 r^2 + r^2 + r r_0 + r_0^2)}{r_0(r+r_0)r}\right)\left(\left(\frac{r}{r_0}\right)^2 - 1\right)^{-\frac{1}{2}}\right| - \pi \\ &\simeq \frac{4\mu(1+\kappa_2 a_0)}{r_0}\end{aligned}$$

Gravitational Lensing:

So:

$$\begin{aligned}\Delta\phi &= 2|\phi(r_0) - \phi_\infty| - \pi \\ &\simeq 2\left|\int_{r_0}^{\infty}\frac{dr}{r}\left(1 + \frac{\mu(2\kappa_2 a_0 r^2 + r^2 + r r_0 + r_0^2)}{r_0(r+r_0)r}\right)\left(\left(\frac{r}{r_0}\right)^2 - 1\right)^{-\frac{1}{2}}\right| - \pi \\ &\simeq \frac{4\mu(1+\kappa_2 a_0)}{r_0}\end{aligned}$$

We have an experimental value $\Delta\phi_{Exp} = 1.761'' \pm 0.016''$ for the sun[2]. So:

$$|\kappa_2 a_0| < 0.009086.$$

[2] See E.B. Formalont and R.A. Sramek, Physical Review Letter, 36, 1475, (1976).

Gravitational Lensing:

So:

$$\begin{aligned}\Delta\phi &= 2|\phi(r_0) - \phi_\infty| - \pi \\ &\simeq 2\left|\int_{r_0}^{\infty} \frac{dr}{r}\left(1 + \frac{\mu(2\kappa_2 a_0 r^2 + r^2 + r r_0 + r_0^2)}{r_0(r + r_0)r}\right)\left(\left(\frac{r}{r_0}\right)^2 - 1\right)^{-\frac{1}{2}}\right| - \pi \\ &\simeq \frac{4\mu(1 + \kappa_2 a_0)}{r_0}\end{aligned}$$

We have an experimental value $\Delta\phi_{Exp} = 1.761'' \pm 0.016''$ for the sun. So:

$$|\kappa_2 a_0| < 0.009086.$$

The deflection of light allow an additional mass given by $M_{add} = \kappa_2 a_0 M$, where M is the mass of sun. If we accept that this mass is dark matter, we have $< 1\%$ of dark matter in the solar system scale. However, in a galactic scale, this effect could be even bigger.

Non-Relativistic Case.

Newtonian Limit:

To express this approximation, we must use:

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\phi\epsilon^2)c^2 & 0 & 0 & 0 \\ 0 & 1-2\phi\epsilon^2 & 0 & 0 \\ 0 & 0 & 1-2\phi\epsilon^2 & 0 \\ 0 & 0 & 0 & 1-2\phi\epsilon^2 \end{pmatrix}$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -2\tilde{\phi}\epsilon^2 c^2 & 0 & 0 & 0 \\ 0 & -2\tilde{\phi}\epsilon^2 & 0 & 0 \\ 0 & 0 & -2\tilde{\phi}\epsilon^2 & 0 \\ 0 & 0 & 0 & -2\tilde{\phi}\epsilon^2 \end{pmatrix}$$

where $\phi = \phi(x,y,z)$ and $\tilde{\phi} = \tilde{\phi}(x,y,z)$ are gravitational potentials and $\epsilon \sim \frac{v}{c}$ is the perturbative parameter. We have used that $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ and $\tilde{g}_{\mu\nu} \rightarrow 0$ for $r \rightarrow \infty$.

Newtonian Limit:

To express this approximation, we must use:

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\phi\epsilon^2)c^2 & 0 & 0 & 0 \\ 0 & 1-2\phi\epsilon^2 & 0 & 0 \\ 0 & 0 & 1-2\phi\epsilon^2 & 0 \\ 0 & 0 & 0 & 1-2\phi\epsilon^2 \end{pmatrix}$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -2\tilde{\phi}\epsilon^2 c^2 & 0 & 0 & 0 \\ 0 & -2\tilde{\phi}\epsilon^2 & 0 & 0 \\ 0 & 0 & -2\tilde{\phi}\epsilon^2 & 0 \\ 0 & 0 & 0 & -2\tilde{\phi}\epsilon^2 \end{pmatrix}$$

where $\phi = \phi(x,y,z)$ and $\tilde{\phi} = \tilde{\phi}(x,y,z)$ are gravitational potentials and $\epsilon \sim \frac{v}{c}$ is the perturbative parameter. So, the equations of motion are reduced to:

$$\partial^2\phi = \frac{\hat{\kappa}}{2}\rho$$

$$\partial^2\tilde{\phi} = \frac{\hat{\kappa}}{2}\tilde{\rho}$$

Besides, $\dot{\rho} = \dot{\tilde{\rho}} = 0$. Therefore, we do not have a relation between ρ and $\tilde{\rho}$.

Post-Newtonian Limit:

If we introduce one order more for the Newtonian limit:

$$
\begin{aligned}
g_{\mu\nu}dx^{\mu}dx^{\nu} \quad = \quad & -\left(1+2\phi\epsilon^{2}+2\left(\phi^{2}+\psi\right)\epsilon^{4}\right)\left(\frac{cdt}{\epsilon}\right)^{2} \\
& +\left(1-2\phi\epsilon^{2}-2\psi\epsilon^{4}\right)\left(dx^{2}+dy^{2}+dz^{2}\right) \\
& +2\epsilon^{3}\left(\chi_{1}dx+\chi_{2}dy+\chi_{3}dz\right)\left(\frac{cdt}{\epsilon}\right) \\
& +\epsilon^{4}\left(\xi_{11}dx^{2}+\xi_{22}dy^{2}+\xi_{33}dz^{2}+2\xi_{12}dxdy+2\xi_{13}dxdz\right. \\
& \left.+2\xi_{23}dydz\right)
\end{aligned}
$$

Post-Newtonian Limit:

If we introduce one order more for the Newtonian limit:

$$\begin{aligned}
\tilde{g}_{\mu\nu}dx^{\mu}dx^{\nu} &= -2\left(\tilde{\phi}\epsilon^2 + \left(2\phi\tilde{\phi}+\tilde{\psi}\right)\epsilon^4\right)\left(\frac{cdt}{\epsilon}\right)^2 \\
&\quad -2\left(\tilde{\phi}\epsilon^2 + \tilde{\psi}\epsilon^4\right)\left(dx^2+dy^2+dz^2\right) \\
&\quad +2\epsilon^3\left(\tilde{\chi}_1 dx + \tilde{\chi}_2 dy + \tilde{\chi}_3 dz\right)\left(\frac{cdt}{\epsilon}\right) \\
&\quad +\epsilon^4\left(\tilde{\xi}_{11}dx^2 + \tilde{\xi}_{22}dy^2 + \tilde{\xi}_{33}dz^2 + 2\tilde{\xi}_{12}dxdy + 2\tilde{\xi}_{13}dxdz \right. \\
&\quad \left. +2\tilde{\xi}_{23}dydz\right)
\end{aligned}$$

All functions depend of (t,x,y,z), but $\frac{1}{c}\frac{\partial}{\partial t} \sim \epsilon$. For this reason, we use $ct \to \frac{ct}{\epsilon}$ to obtain the equations. Besides, we need to fix the gauge.

Post-Newtonian Limit:

The extended harmonic gauge say us:

$$
\begin{aligned}
4\dot{\phi} + \partial_i \chi_i &= 0 \\
2\phi\partial_i\phi - \dot{\chi}_i - \frac{1}{2}\partial_i \xi_{jj} + \partial_j \xi_{ij} &= 0 \\
4\dot{\tilde{\phi}} + \partial_i \tilde{\chi}_i &= 0 \\
2\phi\partial_i\tilde{\phi} + 2\tilde{\phi}\partial_i\phi - \dot{\tilde{\chi}}_i - \frac{1}{2}\partial_i \tilde{\xi}_{jj} + \partial_j \tilde{\xi}_{ij} &= 0
\end{aligned}
$$

Post-Newtonian Limit:

The extended harmonic gauge say us, with a perfect fluid, that the equations of motion are reduced to:

$$
\begin{aligned}
\partial^2\phi &= \frac{\hat{\kappa}}{2}\rho^{(0)} \\
\partial^2\chi_i &= -2\hat{\kappa}U_i^{(1)}\rho^{(0)} \\
\partial^2\psi &= \frac{\hat{\kappa}}{2}\left(2\left(U_k^{(1)}U_k^{(1)}-\phi\right)\rho^{(0)}+\rho^{(2)}+3p^{(2)}(\rho)\right)+\ddot{\phi} \\
\partial^2\xi_{ij} &= -2\hat{\kappa}U_i^{(1)}U_j^{(1)}\rho^{(0)}-4(\partial_i\phi)(\partial_j\phi) \\
&\quad +2\hat{\kappa}\left(\left(U_k^{(1)}U_k^{(1)}+\phi\right)\rho^{(0)}+2p^{(2)}(\rho)\right)\delta_{ij}+4(\partial_k\phi)(\partial_k\phi)\delta_{ij}
\end{aligned}
$$

Post-Newtonian Limit:

The extended harmonic gauge say us, with a perfect fluid, that the equations of motion are reduced to:

$$
\begin{aligned}
\partial^2\tilde{\phi} &= \frac{\hat{\kappa}}{2}\tilde{\rho}^{(0)} \\
\partial^2\tilde{\chi}_i &= -2\hat{\kappa}\left(U_i^{T(1)}\rho^{(0)} + U_i^{(1)}\tilde{\rho}^{(0)}\right) \\
\partial^2\tilde{\psi} &= \hat{\kappa}\left(\left(2U_k^{(1)}U_k^{T(1)} - \tilde{\phi}\right)\rho^{(0)} + \left(U_k^{(1)}U_k^{(1)} - \phi + \frac{3}{2}p'^{(2)}(\rho)\right)\tilde{\rho}^{(0)}\right) \\
&\quad + \frac{\hat{\kappa}}{2}\tilde{\rho}^{(2)} + \ddot{\tilde{\phi}} \\
\partial^2\tilde{\xi}_{ij} &= -2\hat{\kappa}\left(\left(U_i^{T(1)}U_j^{(1)} + U_i^{(1)}U_j^{T(1)}\right)\rho^{(0)} + U_i^{(1)}U_j^{(1)}\tilde{\rho}^{(0)}\right) \\
&\quad -4(\partial_i\tilde{\phi})(\partial_j\phi) - 4(\partial_i\phi)(\partial_j\tilde{\phi}) \\
&\quad +2\hat{\kappa}\left(\left(2U_k^{(1)}U_k^{T(1)} + \tilde{\phi}\right)\rho^{(0)} + \left(U_k^{(1)}U_k^{(1)} + \phi + 2p'^{(2)}(\rho)\right)\tilde{\rho}^{(0)}\right)\delta_{ij} \\
&\quad +8(\partial_k\phi)(\partial_k\tilde{\phi})\delta_{ij}
\end{aligned}
$$

Where $p'^{(2)}(\rho) = \frac{\partial p^{(2)}}{\partial \rho}(\rho)$.

Post-Newtonian Limit:

Besides, we have the conservation equations, but they are null with the gauge equations. However, it is useful to write them in terms of $\rho^{(0)}$, $\rho^{(2)}$, $\tilde{\rho}^{(0)}$, $\tilde{\rho}^{(2)}$ and $p^{(2)}$ in the case when $U_i^{(1)} = U_i^{T(1)} = 0$.

Post-Newtonian Limit:

Besides, we have the conservation equations, but they are null with the gauge equations. However, it is useful to write them in terms of $\rho^{(0)}$, $\rho^{(2)}$, $\tilde{\rho}^{(0)}$, $\tilde{\rho}^{(2)}$ and $p^{(2)}$ in the case when $U_i^{(1)} = U_i^{T(1)} = 0$. That is:

$$\begin{aligned}
\dot{\rho}^{(0)} &= 0 \\
\dot{\rho}^{(2)} &= 0 \\
\dot{\tilde{\rho}}^{(0)} &= 0 \\
\dot{\tilde{\rho}}^{(2)} &= 0 \\
\partial_i p^{(2)}(\rho) &= -\rho^{(0)}\partial_i\phi \\
\partial_i\left(p'^{(2)}(\rho)\tilde{\rho}^{(0)}\right) &= -\rho^{(0)}\partial_i\tilde{\phi} - \tilde{\rho}^{(0)}\partial_i\phi
\end{aligned}$$

These equations give us additional information about $\rho^{(0)}$ that we did not have in the Newtonian approximation. This information come from the gauge fixing. To see this explicitly, we will analyze the case with spherical symmetry.

Post-Newtonian Limit:

All densities are t-independent, therefore they are only dependent on r. So:

$$p'^{(2)}\left(\rho(r)\right)\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right) = -\rho^{(0)}(r)\left(\frac{\partial\phi}{\partial r}(r)\right)$$

$$\frac{\partial}{\partial r}\left(p'^{(2)}\left(\rho(r)\right)\tilde{\rho}^{(0)}(r)\right) = -\rho^{(0)}(r)\left(\frac{\partial\tilde{\phi}}{\partial r}(r)\right) - \tilde{\rho}^{(0)}(r)\left(\frac{\partial\phi}{\partial r}(r)\right)$$

Where we have used that $\left(\frac{\partial p^{(2)}}{\partial r}(r)\right) = p'^{(2)}\left(\rho(r)\right)\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right)$.

Post-Newtonian Limit:

All densities are t-independent, therefore they are only dependent on r. So:

$$p'^{(2)}(\rho(r))\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right) = -\rho^{(0)}(r)\left(\frac{\partial\phi}{\partial r}(r)\right) \quad (35)$$

$$\frac{\partial}{\partial r}\left(p'^{(2)}(\rho(r))\,\tilde{\rho}^{(0)}(r)\right) = -\rho^{(0)}(r)\left(\frac{\partial\tilde{\phi}}{\partial r}(r)\right) - \tilde{\rho}^{(0)}(r)\left(\frac{\partial\phi}{\partial r}(r)\right) \quad (36)$$

Where we have used that $\left(\frac{\partial p^{(2)}}{\partial r}(r)\right) = p'^{(2)}(\rho(r))\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right)$. Now, if we combine (35) and (36), we obtain:

$$\tilde{\rho}^{(0)}(r) = \frac{\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right)}{\left(\frac{\partial\phi}{\partial r}(r)\right)}\left(\tilde{\phi}(r) + \tilde{\phi}_0\right) \quad (37)$$

Where $\tilde{\phi}_0$ is an integration constant. This means that we can obtain an expression for $\tilde{\rho}^{(0)}$ if we know $\rho^{(0)}$.

Post-Newtonian Limit:

Finally, we can study the Trajectory of a Particle. The acceleration is given by:

$$\begin{aligned}
\frac{1}{c^2}\frac{d^2\vec{x}}{dt^2} &= -\epsilon^2\nabla\left(\phi_N + \left(2\phi_N^2 + \psi_N\right)\epsilon^2\right) \\
&+\epsilon^4\left(3\vec{v}\dot{\phi}_N + 4\vec{v}\left(\vec{v}\cdot\nabla\phi_N\right) - v^2\nabla\phi_N - \dot{\vec{\chi}}_N + \left(\vec{v}\times\nabla\times\vec{\chi}_N\right)\right) \\
&+\frac{\epsilon^4 k_2^2}{2}\nabla\tilde{\phi}^2 + O\left(\epsilon^6\right)
\end{aligned}$$

Where $\vec{v} = \frac{d\vec{x}}{dt}$, $\phi_N = \phi + \kappa_2\tilde{\phi}$ and analogous expressions for the others fields. We can conclude:

Post-Newtonian Limit:

Finally, we can study the Trajectory of a Particle. The acceleration is given by:

$$\frac{1}{c^2}\frac{d^2\vec{x}}{dt^2} = -\epsilon^2\nabla\left(\phi_N + \left(2\phi_N^2 + \psi_N\right)\epsilon^2\right) + \epsilon^4\left(3\vec{v}\dot{\phi}_N + 4\vec{v}\left(\vec{v}\cdot\nabla\phi_N\right) - v^2\nabla\phi_N - \dot{\vec{\chi}}_N + \left(\vec{v}\times\nabla\times\vec{\chi}_N\right)\right) + \frac{\epsilon^4 k_2^2}{2}\nabla\tilde{\phi}^2 + O\left(\epsilon^6\right)$$

Where $\vec{v} = \frac{d\vec{x}}{dt}$, $\phi_N = \phi + \kappa_2\tilde{\phi}$ and analogous expressions for the others fields. We can conclude:

- We can see that $\frac{1}{c^2}\frac{d^2\vec{x}}{dt^2} = -\epsilon^2\nabla\phi_N$ in the Newtonian limit, so ϕ_N is the effective potential.

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Finally, we can study the Trajectory of a Particle. The acceleration is given by:

$$\begin{aligned}\frac{1}{c^2}\frac{d^2\vec{x}}{dt^2} &= -\epsilon^2\nabla\left(\phi_N + \left(2\phi_N^2 + \psi_N\right)\epsilon^2\right) \\ &+\epsilon^4\left(3\vec{v}\dot{\phi}_N + 4\vec{v}\left(\vec{v}\cdot\nabla\phi_N\right) - v^2\nabla\phi_N - \dot{\vec{\chi}}_N + \left(\vec{v}\times\nabla\times\vec{\chi}_N\right)\right) \\ &+\frac{\epsilon^4 k_2^2}{2}\nabla\tilde{\phi}^2 + O\left(\epsilon^6\right)\end{aligned}$$

Where $\vec{v} = \frac{d\vec{x}}{dt}$, $\phi_N = \phi + \kappa_2\tilde{\phi}$ and analogous expressions for the others fields. We can conclude:

- We can see that $\frac{1}{c^2}\frac{d^2\vec{x}}{dt^2} = -\epsilon^2\nabla\phi_N$ in the Newtonian limit, so ϕ_N is the effective potential.
- Acceleration is similar to the usual case with $\phi \to \phi_N$, with the exception of the last term. It is an attractive contribution.

Post-Newtonian Limit:

Finally, we can study the Trajectory of a Particle. The acceleration is given by:

$$\begin{aligned}\frac{1}{c^2}\frac{d^2\vec{x}}{dt^2} &= -\epsilon^2\nabla\left(\phi_N + \left(2\phi_N^2+\psi_N\right)\epsilon^2\right) \\ &+\epsilon^4\left(3\vec{v}\dot{\phi}_N + 4\vec{v}\left(\vec{v}\cdot\nabla\phi_N\right) - v^2\nabla\phi_N - \dot{\vec{\chi}}_N + \left(\vec{v}\times\nabla\times\vec{\chi}_N\right)\right) \\ &+\frac{\epsilon^4 k_2^2}{2}\nabla\tilde{\phi}^2 + O\left(\epsilon^6\right)\end{aligned}$$

Where $\vec{v}=\frac{d\vec{x}}{dt}$, $\phi_N=\phi+\kappa_2\tilde{\phi}$ and analogous expressions for the others fields. In spherical symmetry:

$$\rho_{eff}(r)=\rho^{(0)}(r)+\kappa_2\frac{\left(\frac{\partial\rho^{(0)}}{\partial r}(r)\right)}{\left(\frac{\partial\phi}{\partial r}(r)\right)}\left(\tilde{\phi}(r)+\tilde{\phi}_0\right)$$

Therefore, we have an additional mass, that could be identify with dark matter.

Density Profiles:

The equation of motion can be written like:

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial \phi(x)}{\partial x}\right)\right) = \frac{\kappa R^2}{2}\rho(x)$$

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial \tilde{\phi}(x)}{\partial x}\right)\right) = \frac{\kappa R^2}{2}\frac{\left(\frac{\partial \rho}{\partial x}(x)\right)}{\left(\frac{\partial \phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1 + 2\epsilon^2\phi(x)\right)\right),$$

where $x = \frac{r}{R}$.

Density Profiles:

The equation of motion can be written like:

$$\begin{aligned}\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) &= \frac{\kappa R^2}{2}\rho(x)\\ \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) &= \frac{\kappa R^2}{2}\frac{\left(\frac{\partial\rho}{\partial x}(x)\right)}{\left(\frac{\partial\phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1+2\epsilon^2\phi(x)\right)\right),\end{aligned}$$

where $x = \frac{r}{R}$. Then, we can define the ordinary and tilde mass respectively as:

$$\begin{aligned}m(x) &\equiv 4\pi R^3\int_0^\infty dx x^2\rho(x) = \frac{8\pi R}{\kappa}x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\\ \tilde{m}(x) &\equiv 4\pi R^3\int_0^\infty dx x^2\tilde{\rho}(x) = \frac{8\pi R}{\kappa}x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right).\end{aligned}$$

Density Profiles:

The equation of motion can be written like:

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) = \frac{\kappa R^2}{2}\rho(x) \quad (38)$$

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) = \frac{\kappa R^2}{2}\frac{\left(\frac{\partial\rho}{\partial x}(x)\right)}{\left(\frac{\partial\phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1+2\epsilon^2\phi(x)\right)\right), \quad (39)$$

where $x = \frac{r}{R}$. Then, we can define the ordinary and tilde mass respectively as:

$$m(x) \equiv 4\pi R^3\int_0^\infty dx x^2\rho(x) = \frac{8\pi R}{\kappa}x^2\left(\frac{\partial\phi(x)}{\partial x}\right) \quad (40)$$

$$\tilde{m}(x) \equiv 4\pi R^3\int_0^\infty dx x^2\tilde{\rho}(x) = \frac{8\pi R}{\kappa}x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right). \quad (41)$$

Finally, defining the total mass $M(x) = m(x) + \kappa_2\tilde{m}(x)$, the rotation velocity is:

$$\left(\frac{v_{rot}(x)}{c}\right)^2 \equiv \epsilon^2 x\frac{\partial}{\partial x}\left(\phi(x)+\kappa_2\tilde{\phi}(x)\right) = \frac{\epsilon^2\kappa M(x)}{8\pi R x}. \quad (42)$$

Density Profiles: Spherically Homogeneous

With $\rho(x) = \rho_0 \Theta(1-x)$, the equations are:

$$\begin{aligned}
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) &= \frac{\kappa R^2 \rho_0}{2}\Theta(1-x) \\
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) &= -\frac{\kappa R^2 \rho_0}{2}\frac{\delta(1-x)}{\left(\frac{\partial\phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1+2\epsilon^2\phi(x)\right)\right).
\end{aligned}$$

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With $\rho(x) = \rho_0 \Theta(1-x)$, the equations are:

$$\begin{aligned} \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial \phi(x)}{\partial x}\right)\right) &= \frac{\kappa R^2 \rho_0}{2}\Theta(1-x) \\ \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial \tilde{\phi}(x)}{\partial x}\right)\right) &= -\frac{\kappa R^2 \rho_0}{2}\frac{\delta(1-x)}{\left(\frac{\partial \phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1+2\epsilon^2\phi(x)\right)\right). \end{aligned}$$

$$\phi(x) = \begin{cases} \frac{\kappa R^2 \rho_0}{12}\left(x^2 - 3\right) & x \leq 1 \\ -\frac{\kappa R^2 \rho_0}{6x} & x > 1 \end{cases} \tag{43}$$

$$\tilde{\phi}(x) = \begin{cases} \frac{3}{2}C\left(1 - \frac{\epsilon^2 \kappa R^2 \rho_0}{3}\right) & x \leq 1 \\ \frac{3}{2x}C\left(1 - \frac{\epsilon^2 \kappa R^2 \rho_0}{3}\right) & x > 1 \end{cases}. \tag{44}$$

On the other side, the force is $\vec{F} = -\epsilon^2 \frac{mc^2}{R}\frac{\partial}{\partial x}\left(\phi(x) + \kappa_2\tilde{\phi}(x)\right)$. We expect that $\vec{F}$ must be continuous, therefore $C = 0$!!! In conclusion, if the ordinary matter has a spherically homogeneous distribution, we do not have $\tilde{\delta}$ matter.

Density Profiles: Exponential

With $\rho(x) = \rho_0 e^{-x}$, the equations are:

$$\begin{aligned}
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) &= \frac{\kappa R^2 \rho_0 e^{-x}}{2} \\
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) &= -\frac{\kappa R^2 \rho_0 e^{-x}}{2\left(\frac{\partial\phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1 + 2\epsilon^2\phi(x)\right)\right).
\end{aligned}$$

Density Profiles: Exponential

With $\rho(x) = \rho_0 e^{-x}$, the equations are:

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) = \frac{\kappa R^2\rho_0 e^{-x}}{2}$$

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) = -\frac{\kappa R^2\rho_0 e^{-x}}{2\left(\frac{\partial\phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1 + 2\epsilon^2\phi(x)\right)\right).$$

$$\phi(x) = \frac{\kappa R^2\rho_0\left(\left(x+2\right)e^{-x} - 2\right)}{2x}$$

$$\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) = -\frac{x^4 e^{-x}\left(\tilde{\phi}(x) - C\left(1 - \frac{\epsilon^2\kappa R^2\rho_0}{x}\left(2 - \left(x+2\right)e^{-x}\right)\right)\right)}{\left(2 - \left(x^2 + 2x + 2\right)e^{-x}\right)}$$

Density Profiles: Exponential

With $\rho(x) = \rho_0 e^{-x}$, the equations are:

$$\begin{aligned}
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) &= \frac{\kappa R^2\rho_0 e^{-x}}{2}\\
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) &= -\frac{\kappa R^2\rho_0 e^{-x}}{2\left(\frac{\partial\phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1+2\epsilon^2\phi(x)\right)\right).
\end{aligned}$$

$$\begin{aligned}
\phi(x) &= \frac{\kappa R^2\rho_0\left(\left(x+2\right)e^{-x}-2\right)}{2x}\\
\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) &= -\frac{x^4 e^{-x}\left(\tilde{\phi}(x) - C\left(1-\frac{\epsilon^2\kappa R^2\rho_0}{x}\left(2-\left(x+2\right)e^{-x}\right)\right)\right)}{\left(2-\left(x^2+2x+2\right)e^{-x}\right)}
\end{aligned}$$

$$m(x) \sim \frac{4\pi R^3\rho_0}{3}x^3 \qquad \tilde{m}(x) \sim \epsilon^2 C\pi R^3\rho_0 x^4. \tag{45}$$

Density Profiles: Exponential

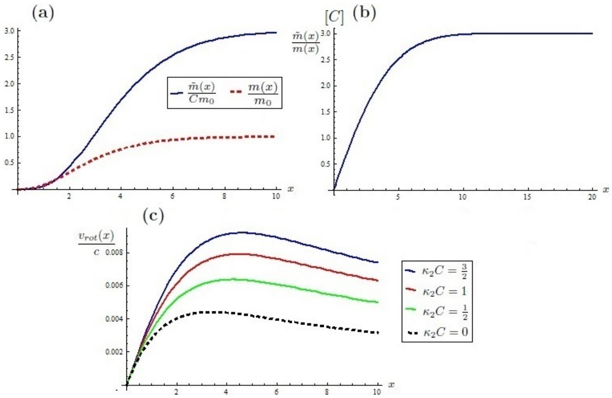

Figure: Exponential Profile Calculation. (a) Ordinary and $\tilde{\delta}$ mass vs normalize radius, with $m_0 = 8\pi R^3 \rho_0$. The initial conditions have been chosen such that $\tilde{m}(x) > 0$ with $C > 0$. (b) $\frac{\tilde{m}(x)}{m(x)}$ (in units of C) vs normalize radius. In the beginning the relation is almost linear, $\frac{\tilde{m}(x)}{m(x)} \sim x$, but in the end it is like a constant, $\frac{\tilde{m}(x)}{m(x)} \to 3C$. (c) Rotation velocity vs normalize radius for different values of C. The Black-Dashed line corresponds to GR case, so the C value indicates the contribution of $\tilde{\delta}$ matter. In these calculations we have used $\kappa\rho_0 R^2 = 10^{-4}$.

Density Profiles: Einasto

With $\rho(x) = \rho_0 e^{-x^\alpha}$, the equations are:

$$\begin{aligned}
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) &= \frac{\kappa R^2 \rho_0 e^{-x^\alpha}}{2} \\
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) &= -\frac{\kappa R^2 \rho_0 \alpha x^{\alpha-1} e^{-x^\alpha}}{2\left(\frac{\partial\phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1+2\epsilon^2\phi(x)\right)\right).
\end{aligned}$$

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With $\rho(x) = \rho_0 e^{-x^\alpha}$, the equations are:

$$\begin{aligned}
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) &= \frac{\kappa R^2\rho_0 e^{-x^\alpha}}{2} \\
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) &= -\frac{\kappa R^2\rho_0\alpha x^{\alpha-1}e^{-x^\alpha}}{2\left(\frac{\partial\phi}{\partial x}(x)\right)}\left(\tilde{\phi}(x) - C\left(1+2\epsilon^2\phi(x)\right)\right).
\end{aligned}$$

$$m(x) \sim \frac{4\pi R^3\rho_0}{3}x^3 \quad (46)$$

$$\tilde{m}(x) \sim \frac{4\pi R^3\rho_0\epsilon^2 C\alpha}{(3+\alpha)}x^{\alpha+3}. \quad (47)$$

Density Profiles: Einasto

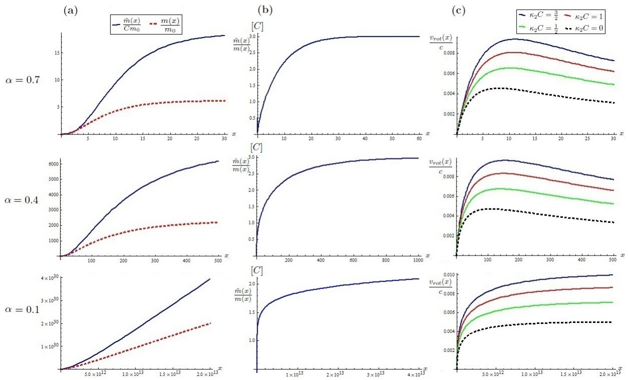

Figure: Einasto Profile Calculation for $\alpha = \{0.7, 0.4, 0.1\}$. (a) Ordinary and $\tilde{\delta}$ mass vs normalize radius, with $m_0 = 8\pi R^3 \rho_0$. The initial conditions have been chosen such that $\tilde{m}(x) > 0$ with $C > 0$. (b) $\frac{\tilde{m}(x)}{m(x)}$ (in units of C) vs normalize radius. (c) Rotation velocity vs normalize radius for different values of C. The Black-Dashed line corresponds to GR case, so the C value indicates the contribution of $\tilde{\delta}$ matter. In these calculations we have used $\kappa \rho_0 R^2 = 4 \left(\frac{\alpha}{2}\right)^{\frac{2}{\alpha}} \times 10^{-4} e^{\frac{2(1-\alpha)}{\alpha}}$.

Density Profiles: Navarro-Frenk-White

With $\rho(x) = \frac{\rho_0}{x^\gamma (x+1)^{3-\gamma}}$, the equations are:

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) = \frac{\kappa R^2 \rho_0}{2x^\gamma (x+1)^{3-\gamma}}$$

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) = -\frac{\kappa R^2 \rho_0 (\gamma+3x)\left(\tilde{\phi}(x) - C\left(1+2\epsilon^2\phi(x)\right)\right)}{2x^{\gamma+1}(x+1)^{4-\gamma}\left(\frac{\partial\phi}{\partial x}(x)\right)}.$$

Density Profiles: Navarro-Frenk-White

With $\rho(x) = \frac{\rho_0}{x^\gamma (x+1)^{3-\gamma}}$, the equations are:

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\phi(x)}{\partial x}\right)\right) = \frac{\kappa R^2 \rho_0}{2x^\gamma (x+1)^{3-\gamma}}$$

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\left(\frac{\partial\tilde{\phi}(x)}{\partial x}\right)\right) = -\frac{\kappa R^2 \rho_0 (\gamma+3x)\left(\tilde{\phi}(x) - C\left(1+2\epsilon^2\phi(x)\right)\right)}{2x^{\gamma+1}(x+1)^{4-\gamma}\left(\frac{\partial\phi}{\partial x}(x)\right)}.$$

$$m(x) \sim \frac{4\pi R^3 \rho_0}{3-\gamma} x^{3-\gamma} \quad (48)$$

$$\tilde{m}(x) \sim \frac{4\pi\epsilon^2 C R^3 \rho_0 \gamma}{3-\gamma} x^{3-\gamma} \quad (49)$$

Density Profiles: Navarro-Frenk-White

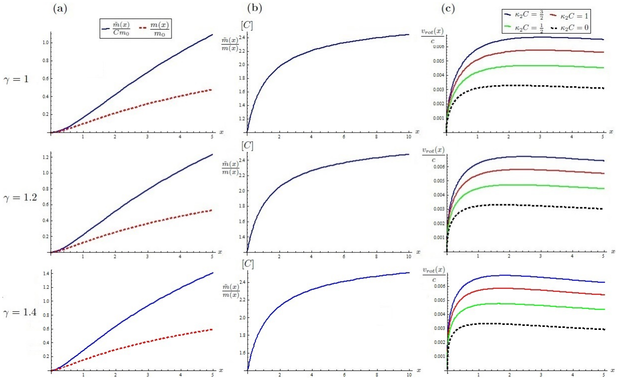

Figure: Navarro-Frenk-White Profile Calculation for $\gamma = \{1, 1.2, 1.4\}$. (a) Ordinary and $\tilde{\delta}$ mass vs normalize radius, with $m_0 = 8\pi R^3 \rho_0$. The initial conditions have been chosen such that $\tilde{m}(x) > 0$ with $C > 0$. (b) $\frac{\tilde{m}(x)}{m(x)}$ (in units of C) vs normalize radius. (c) Rotation velocity vs normalize radius for different values of C. The Black-Dashed line correspond to GR case and the others C values indicate the contribution of $\tilde{\delta}$ matter. In these calculations we have used $\kappa \rho_0 R^2 = 10^{-4} \frac{(3-\gamma)^{3-\gamma}}{4(2-\gamma)^{2-\gamma}}$.

Cosmological Case.

FRW and Photon Trajectory:

We have that:

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2(\theta) \end{pmatrix}$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -c^2F_b(t) & 0 & 0 & 0 \\ 0 & F_a(t)a^2(t) & 0 & 0 \\ 0 & 0 & F_a(t)a^2(t)r^2 & 0 \\ 0 & 0 & 0 & F_a(t)a^2(t)r^2\sin^2(\theta) \end{pmatrix}$$

we want analyze the trajectory of a supernova photon when it is traveling to the Earth. For this we use a radial trajectory from r_1 to $r = 0$.

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$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -c^2 F_b(t) & 0 & 0 & 0 \\ 0 & F_a(t)a^2(t) & 0 & 0 \\ 0 & 0 & F_a(t)a^2(t)r^2 & 0 \\ 0 & 0 & 0 & F_a(t)a^2(t)r^2\sin^2(\theta) \end{pmatrix} \tag{51}$$

we want analyze the trajectory of a supernova photon when it is traveling to the Earth. For this we use a radial trajectory from r_1 to $r = 0$. So, we have:

$$-(1 + \kappa_2 F_b(t))c^2 dt^2 + a^2(t)(1 + \kappa_2 F_a(t))dr^2 = 0$$

We interpret that like a modified scale factor: $a_{eff}(t) = a(t)\sqrt{\frac{1+\kappa_2 F_a(t)}{1+\kappa_2 F_b(t)}}$.

Redshift and Luminosity Distance:

If we integrate this expression from r_1 to 0, we obtain:

$$r_1 = c \int_{t_1}^{t_0} \frac{dt}{a_{eff}(t)}$$

Where t_1 and t_0 are the emission and reception time.

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Where t_1 and t_0 are the emission and reception time. If a second wave crest is emitted at $t = t_1 + \Delta t_1$ from $r = r_1$, it will reach $r = 0$ at $t = t_0 + \Delta t_0$, so:

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Therefore, for Δt_1, Δt_0 small, which is appropriate for light waves, we get:

$$\frac{\Delta t_0}{\Delta t_1} = \frac{a_{eff}(t_0)}{a_{eff}(t_1)}$$

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Therefore, for Δt_1, Δt_0 small, which is appropriate for light waves, we get:

$$\frac{\Delta t_0}{\Delta t_1} = \frac{a_{eff}(t_0)}{a_{eff}(t_1)}$$

Since t is the proper time, the redshift is now:

$$1 + z(t_1) = \frac{a_{eff}\,(t_0)}{a_{eff}(t_1)}.$$

Redshift and Luminosity Distance:

If we integrate this expression from r_1 to 0, we obtain:

$$r_1 = c\int_{t_1}^{t_0} \frac{dt}{a_{eff}(t)}$$

Where t_1 and t_0 are the emission and reception time. Then, the redshift is:

$$1 + z(t_1) = \frac{a_{eff}(t_0)}{a_{eff}(t_1)} \tag{52}$$

and the luminosity distance is given by:

$$\begin{aligned} d_L &= \frac{a_{eff}^2(t_0)}{a_{eff}(t_1)} r_1 \\ &= c\frac{a_{eff}^2(t_0)}{a_{eff}(t_1)} \int_{t_1}^{t_0} \frac{dt}{a_{eff}(t)} \end{aligned} \tag{53}$$

Einstein's Equations: Exact Solution

Since we wish to explain DE with $\tilde{\delta}$ gravity, we will assume that the universe only have non relativistic matter (cold dark matter, baryonic matter) and radiation (photons, massless particles). For non relativistic matter we use $p_M(t) = 0$ and for radiation $p_R(t) = \frac{1}{3}\rho_R(t)$.

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$$\rho(Y) = \frac{3H_0^2\Omega_R}{\kappa c^2 C}\frac{Y+C}{Y^4} \quad (54)$$

$$p(Y) = \frac{H_0^2\Omega_R}{\kappa c^2}\frac{1}{Y^4} \quad (55)$$

$$t(Y) = \frac{2\sqrt{C}}{3H_0\sqrt{\Omega_R}}\left(\sqrt{Y+C}(Y-2C)+2C^{\frac{3}{2}}\right) \quad (56)$$

$$Y = \frac{a(t)}{a_0} \quad (57)$$

Where $t(Y)$ is the time variable, a_0 is the scale factor in the present, $C = \frac{\Omega_R}{\Omega_M}$, and Ω_R and Ω_M are the radiation and non-relativistic matter density in the present respectively, with $\Omega_M = 1 - \Omega_R$.

New Equations:

The equation of $\tilde{g}_{\mu\nu}$ is:

$$F^{(\mu\nu)(\alpha\beta)\rho\lambda}D_\rho D_\lambda \tilde{g}_{\alpha\beta} + \frac{1}{2}\left(g^{\mu\nu}R^{\alpha\beta}\tilde{g}_{\alpha\beta} - \tilde{g}^{\mu\nu}R\right) = \kappa\tilde{T}^{\mu\nu}$$

$$D_\nu \tilde{T}^{\mu\nu} = \frac{1}{2}T^{\alpha\beta}D^\mu\tilde{g}_{\alpha\beta} - \frac{1}{2}T^{\mu\beta}D_\beta\tilde{g}^\alpha_\alpha + D_\beta(\tilde{g}^\beta_\alpha T^{\alpha\mu})$$

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Besides, we need to fix the gauge. The extended harmonic gauge say us that $F_b(t) = 3F_a(t)$.

New Equations:

The equation of $\tilde{g}_{\mu\nu}$ is:

$$F^{(\mu\nu)(\alpha\beta)\rho\lambda}D_\rho D_\lambda \tilde{g}_{\alpha\beta} + \frac{1}{2}\left(g^{\mu\nu}R^{\alpha\beta}\tilde{g}_{\alpha\beta} - \tilde{g}^{\mu\nu}R\right) = \kappa\tilde{T}^{\mu\nu} \quad (58)$$

$$D_\nu\tilde{T}^{\mu\nu} = \frac{1}{2}T^{\alpha\beta}D^\mu\tilde{g}_{\alpha\beta} - \frac{1}{2}T^{\mu\beta}D_\beta\tilde{g}^\alpha_\alpha + D_\beta(\tilde{g}^\beta_\alpha T^{\alpha\mu}) \quad (59)$$

Besides, we need to fix the gauge. The extended harmonic gauge say us that $F_b(t) = 3F_a(t)$. So, the solution to $\tilde{g}_{\mu\nu}$ is:

$$\tilde{\rho}_M(Y) = \frac{9H_0^2\Omega_R}{2\kappa c^2 C}\frac{(C_1 - T_d(Y))}{Y^3} \quad (60)$$

$$\tilde{\rho}_R(Y) = \frac{6H_0^2\Omega_R}{\kappa c^2}\frac{(C_2 - T_d(Y))}{Y^4} \quad (61)$$

$$F_a(Y) = \frac{3}{2}(2C_2 - C_1)\frac{Y}{C}\left(\sqrt{\frac{Y}{C}+1}\ln\left(\frac{\sqrt{\frac{Y}{C}+1}+1}{\sqrt{\frac{Y}{C}+1}-1}\right) - 2\right) - 2C_2 + C_3\frac{Y}{C}\sqrt{\frac{Y}{C}+1} \quad (62)$$

New Equations:

Using $\tilde{Y} = \frac{a_{eff}(t)}{a(t_0)}$, we can see that, when $Y \ll C$, $\tilde{Y} = \sqrt{\frac{1-2k_2C_2}{1-6k_2C_2}} Y + O(Y^2)$. We want $\tilde{Y} = Y + O(Y^2)$ in this case, because we expect that $\tilde{\delta}$ gravity explain dark energy and it is irrelevant in the early universe. For this, we must use $C_2 = 0$. Besides, we chose the other constants such that a Big-Rip is produced. That is $\tilde{Y}(Y_{Rip}) = \infty$.

New Equations:

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$$\tilde{Y}(Y, L_1, L_2, C) = Y\sqrt{\frac{1 - L_2\frac{Y}{3}\sqrt{Y+C} + L_1\frac{Y}{C}\left(\sqrt{\frac{Y}{C}+1}\ln\left(\frac{\sqrt{\frac{Y}{C}+1}+1}{\sqrt{\frac{Y}{C}+1}-1}\right) - 2\right)}{1 - L_2Y\sqrt{Y+C} + 3L_1\frac{Y}{C}\left(\sqrt{\frac{Y}{C}+1}\ln\left(\frac{\sqrt{\frac{Y}{C}+1}+1}{\sqrt{\frac{Y}{C}+1}-1}\right) - 2\right)}}$$

We need a Big-Rip to explain the accelerated expansion of the universe because we want that $\tilde{Y}$ to grow quickly and simulate the dark energy.

New Equations:

Using $\tilde{Y}=\frac{a_{eff}(t)}{a(t_0)}$, we can see that, when $Y \ll C$, $\tilde{Y}=\sqrt{\frac{1-2k_2C_2}{1-6k_2C_2}}Y+O(Y^2)$. We want $\tilde{Y}=Y+O(Y^2)$ in this case, because we expect that $\tilde{\delta}$ gravity explain dark energy and it is irrelevant in the early universe. For this, we must use $C_2=0$. Besides, we chose the other constants such that a Big-Rip is produced. That is $\tilde{Y}(Y_{Rip})=\infty$. Therefore, the modified scale factor is:

$$\tilde{Y}(Y,L_1,L_2,C)= Y\sqrt{\frac{1-L_2\frac{Y}{3}\sqrt{Y+C}+L_1\frac{Y}{C}\left(\sqrt{\frac{Y}{C}+1}\ln\left(\frac{\sqrt{\frac{Y}{C}+1}+1}{\sqrt{\frac{Y}{C}+1}-1}\right)-2\right)}{1-L_2Y\sqrt{Y+C}+3L_1\frac{Y}{C}\left(\sqrt{\frac{Y}{C}+1}\ln\left(\frac{\sqrt{\frac{Y}{C}+1}+1}{\sqrt{\frac{Y}{C}+1}-1}\right)-2\right)}} \quad (63)$$

From (63), we can see that the Big-Rip is produced when:

$$Y_{Rip}=\left(\frac{1+2L_1}{L_2}\right)^{\frac{2}{3}} \quad (64)$$

Results: Distance Modulus

The distance modulus is:

$$\mu(z) \equiv m(z) - M = 5\log_{10}\left(\frac{d_L(z)}{10 \text{ pc}}\right)$$

where $m(z)$ and M are the apparent magnitude and absolute magnitude respectively. M is common for all supernova, so it is constant. The difference between GR and $\tilde{\delta}$ gravity is in $d_L(z)$. In GR, we have that:

$$d_L(z) = \frac{c(1+z) \text{ Mpc s}}{100 \text{ km}} \int_{\frac{1}{1+z}}^{1} \frac{dY'}{\sqrt{h^2\Omega_\Lambda Y'^4 + h^2\Omega_M Y' + h^2\Omega_R}} \quad (65)$$

With $\Omega_\Lambda = 1 - \Omega_M - \Omega_R$.

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$$d_L(z) \quad = \quad c(1+z)\frac{\sqrt{C}}{H_0\sqrt{\Omega_R}}\int_0^z \frac{(1+u)Y(u)Y'(u)}{\sqrt{Y(u)+C}}du$$

where $Y'(z) = \frac{dY}{dz}(z)$.

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where $Y'(z) = \frac{dY}{dz}(z)$. To find $Y(z)$, we must solve $\tilde{Y}(Y(z)) = \frac{\tilde{Y}_0}{1+z}$, where:

$$\tilde{Y}(Y, L_1, L_2, C) = \qquad (67)$$

$$Y\sqrt{\frac{1 - L_2\frac{Y}{3}\sqrt{Y+C} + L_1\frac{Y}{C}\left(\sqrt{\frac{Y}{C}+1}\ln\left(\frac{\sqrt{\frac{Y}{C}+1}+1}{\sqrt{\frac{Y}{C}+1}-1}\right) - 2\right)}{1 - L_2 Y\sqrt{Y+C} + 3L_1\frac{Y}{C}\left(\sqrt{\frac{Y}{C}+1}\ln\left(\frac{\sqrt{\frac{Y}{C}+1}+1}{\sqrt{\frac{Y}{C}+1}-1}\right) - 2\right)}}.$$

Cosmology in Delta Gravity: A Classical Analysis and Phenomenology.

by Pablo González V.

Results:

Using the supernova data (see N. Suzuki, et al., ApJ, Vol. 746, Number 1, Pp. 85, (2012)), we obtain that:

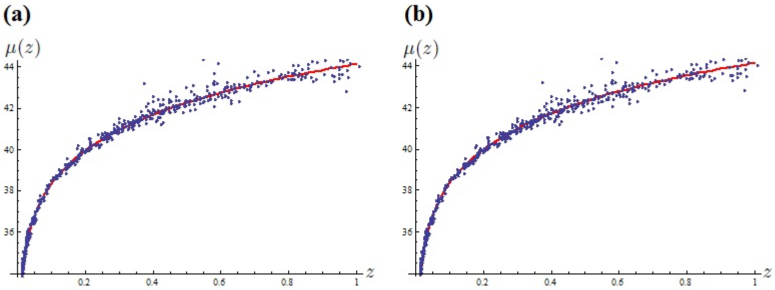

Figure: Distance modulus vs Redshift. We have fitted 580 supernovae to (a) GR and (b) $\tilde{\delta}$ Gravity.

Results:

Using the supernova data (see N. Suzuki, et al., ApJ, Vol. 746, Number 1, Pp. 85, (2012)), we obtain that:

In GR:

$h = 0.7$ and $h^2\Omega_M = 0.136$ with $\chi^2(\text{per point}) = 0.985$.

In $\tilde{\delta}$ gravity:

$L_1 = 1.565$, $L_2 = 2.262$ and $C = 1.82 \times 10^{-4}$ with $\chi^2(\text{per point}) = 0.996$.

We have used that $H_0\sqrt{\Omega_R} = 0.644$ km s^{-1} Mpc^{-1}.

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$$\tilde{\Omega}_M = \left|\frac{\kappa c^2 \kappa_2 \tilde{\rho}_{M0}}{3H_0^2}\right|$$

$$\tilde{\Omega}_R = \left|\frac{\kappa c^2 \kappa_2 \tilde{\rho}_{R0}}{3H_0^2}\right|,$$

[3] We include κ_2 in the definitions of $\tilde{\rho}_{M0}$ and $\tilde{\rho}_{R0}$ because it is just a control parameter. Actually, in all our $\tilde{\delta}$ definitions is possible to "absorb" κ_2 on another parameter. The absolute value is incorporated to guarantee that the normalize $\tilde{\delta}$ densities are positive. In fact, in this case we need $\kappa_2 < 0$.

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$$\tilde{\Omega}_M \approx \frac{|L_2 - 4L_1|}{2}\Omega_M \approx 2\Omega_M \quad (68)$$

$$\tilde{\Omega}_R \approx \frac{2\,|L_2 - 2L_1|}{3}\Omega_R \approx 0.58\Omega_R. \quad (69)$$

More Results:

The time is given by (56):

$$t(Y) = \frac{2\sqrt{C}}{3H_0\sqrt{\Omega_R}}\left(\sqrt{Y+C}(Y-2C)+2C^{\frac{3}{2}}\right)$$

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Age of the Universe (t_0):

That is $Y(t) = 1$. So:
In GR $\rightarrow t_0 = 1.367 \times 10^{10}$ [yr]
In $\tilde{\delta}$ gravity $\rightarrow t_0 = 1.364 \times 10^{10}$ [yr]

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Big Rip time (t_{BR}):

That is $\tilde{Y}(t) = \infty$. So:
In $\tilde{\delta}$ gravity $Y_{Rip} = 1.826 \rightarrow t_{Rip} = 3.366 \times 10^{10}$ [yr]

Inflation.

$\tilde{\delta}$ Inflation:

In inflation, the fluid is represented by a scalar field with an action given by:

$$S_0 = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{1}{2}g^{\alpha\beta}\left(\partial_\alpha\varphi\right)\left(\partial_\beta\varphi\right) - V\left(\varphi\right)\right), \tag{70}$$

where φ is the inflaton and $V(\varphi)$ is the potential of the scalar field.

$\tilde{\delta}$ Inflation:

In $\tilde{\delta}$ Gravity we have an additional scalar field, $\tilde{\varphi}$, then:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{\kappa_2}{2\kappa}G^{\mu\nu}\tilde{g}_{\mu\nu} - \frac{1}{2}\left(1 + \frac{\kappa_2}{2}\tilde{g}^{\lambda}_{\lambda}\right)g^{\alpha\beta}\left(\partial_\alpha\varphi\right)\left(\partial_\beta\varphi\right)\right. \\ &\left. + \frac{\kappa_2}{2}\tilde{g}^{\alpha\beta}\left(\partial_\alpha\varphi\right)\left(\partial_\beta\varphi\right) - \kappa_2 g^{\alpha\beta}\left(\partial_\alpha\varphi\right)\left(\partial_\beta\tilde{\varphi}\right) - \mathbf{V}(\varphi,\tilde{\varphi})\right), \end{aligned}$$

where $\tilde{\varphi}$ is the new inflaton and $\mathbf{V}(\varphi,\tilde{\varphi}) = \left(1 + \frac{\kappa_2}{2}\tilde{g}^{\lambda}_{\lambda}\right)V(\varphi) + \kappa_2 V_{,\varphi}(\varphi)\tilde{\varphi}$ is the effective potential, with $V_{,\varphi}(\varphi) = \frac{\partial V}{\partial\varphi}(\varphi)$.

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where $\tilde{\varphi}$ is the new inflaton and $\mathbf{V}(\varphi,\tilde{\varphi}) = \left(1+\frac{\kappa_2}{2}\tilde{g}^{\lambda}_{\lambda}\right)V(\varphi) + \kappa_2 V_{,\varphi}(\varphi)\tilde{\varphi}$ is the effective potential, with $V_{,\varphi}(\varphi) = \frac{\partial V}{\partial\varphi}(\varphi)$. In the background, we have that $\varphi = \varphi_0(t)$ and $\tilde{\varphi} = \tilde{\varphi}_0(t)$. The cosmological equations are:

$$H^2(t) = \frac{\kappa}{3}\left(\frac{1}{2}\dot{\varphi}_0^2(t) + V(\varphi_0(t))\right) \quad (71)$$

$$\dot{H}(t) = -\frac{\kappa}{2}\dot{\varphi}_0^2(t) \quad (72)$$

$$H(t)\dot{F}_a(t) - V(\varphi_0(t))F_a(t) = \frac{\kappa}{3}\left(\dot{\varphi}_0(t)\dot{\tilde{\varphi}}_0(t) + V_{,\varphi}(\varphi_0(t))\tilde{\varphi}_0(t)\right) \quad (73)$$

$$\ddot{F}_a(t) - 3H(t)\dot{F}_a(t) = -2\kappa\dot{\varphi}_0(t)\dot{\tilde{\varphi}}_0(t). \quad (74)$$

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In $\tilde{\delta}$ Gravity we have an additional scalar field, $\tilde{\varphi}$, then:

$$\begin{aligned} S \; = \; & \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{\kappa_2}{2\kappa}G^{\mu\nu}\tilde{g}_{\mu\nu} - \frac{1}{2}\left(1 + \frac{\kappa_2}{2}\tilde{g}^{\lambda}_{\lambda}\right) g^{\alpha\beta}\left(\partial_\alpha\varphi\right)\left(\partial_\beta\varphi\right) \right. \\ & \left. + \frac{\kappa_2}{2}\tilde{g}^{\alpha\beta}\left(\partial_\alpha\varphi\right)\left(\partial_\beta\varphi\right) - \kappa_2 g^{\alpha\beta}\left(\partial_\alpha\varphi\right)\left(\partial_\beta\tilde{\varphi}\right) - \mathbf{V}(\varphi,\tilde{\varphi})\right), \end{aligned} \quad (75)$$

where $\tilde{\varphi}$ is the new inflaton and $\mathbf{V}(\varphi,\tilde{\varphi}) = \left(1 + \frac{\kappa_2}{2}\tilde{g}^{\lambda}_{\lambda}\right) V(\varphi) + \kappa_2 V_{,\varphi}(\varphi)\tilde{\varphi}$ is the effective potential, with $V_{,\varphi}(\varphi) = \frac{\partial V}{\partial \varphi}(\varphi)$. Additionally, the field equations of motion are:

$$\ddot{\varphi}_0(t) + 3H(t)\dot{\varphi}_0(t) + V_{,\varphi}(\varphi_0(t)) = 0 \quad (76)$$

$$\ddot{\tilde{\varphi}}_0(t) + 3H(t)\dot{\tilde{\varphi}}_0(t) + V_{,\varphi\varphi}(\varphi_0(t))\tilde{\varphi}_0(t) = -3V_{,\varphi}(\varphi_0(t))F_a(t). \quad (77)$$

Inflation Parameters:

We define:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\kappa}{2H^2}\dot{\varphi}_0^2 \tag{78}$$

$$\eta \equiv -\frac{\ddot{\varphi}_0}{H\dot{\varphi}_0} = 3 + \frac{V_\varphi}{H\dot{\varphi}_0} \tag{79}$$

$$\tilde{\epsilon} \equiv \frac{\kappa}{H^2}\dot{\varphi}_0\dot{\tilde{\varphi}}_0 \tag{80}$$

$$\tilde{\eta} \equiv -\frac{\ddot{\varphi}_0}{H\dot{\varphi}_0} = 3\frac{\dot{F}_a}{H} - 3(3-\epsilon)F_a + \frac{\kappa(3-\eta)\dot{\varphi}_0\tilde{\varphi}_0}{H} \tag{81}$$

$$H_{eff} \equiv \frac{\dot{a}_{eff}}{a_{eff}} = H\left(1 - \frac{\kappa_2\dot{F}_a}{H(1+\kappa_2F_a)(1+3\kappa_2F_a)}\right) \tag{82}$$

$$\epsilon_{eff} \equiv -\frac{\dot{H}_{eff}}{H_{eff}^2} = \epsilon + \frac{H(3-\epsilon)}{H_{eff}}\left(\frac{H_{eff}}{H} - 1 + \frac{2\kappa_2\left(3F_a + \frac{V_\varphi}{V}\tilde{\varphi}_0\right)}{(1+\kappa_2F_a)(1+3\kappa_2F_a)}\right) - \frac{2H(2+3\kappa_2F_a)}{H_{eff}}\left(\frac{H_{eff}}{H} - 1\right)^2. \tag{83}$$

Inflation Equations:

Using the number e-folds (N):

$$\frac{d}{dt} = H\frac{d}{dN},$$

Background:

$$H^2 - \frac{\kappa V}{3 - \frac{\kappa}{2}\varphi_0'^2} = 0 \quad (84)$$

$$F_a' - \left(3 - \frac{\kappa}{2}\varphi_0'^2\right)\left(F_a' + \frac{V_{,\varphi}\tilde{\varphi}_0}{3V}\right) - \frac{\kappa}{3}\varphi_0'\tilde{\varphi}_0' = 0 \quad (85)$$

$$\varphi_0'' + \left(3 - \frac{\kappa}{2}\varphi_0'^2\right)\left(\varphi_0' + \frac{V_{,\varphi}}{\kappa V}\right) = 0 \quad (86)$$

$$\tilde{\varphi}_0'' + \left(3 - \frac{\kappa}{2}\varphi_0'^2\right)\left(\tilde{\varphi}_0' + \frac{V_{,\varphi\varphi}\tilde{\varphi} + 3V_{,\varphi}F_a}{\kappa V}\right) = 0. \quad (87)$$

Inflation Equations:

Using the number of e-folds (N):

$$\frac{d}{dt} = H\frac{d}{dN},$$

Perturbation:

$$\mathcal{R}'' + Q^2\mathcal{R} - \left(3 - \frac{\kappa}{2}\varphi_0'^2\right)\left(1 + \frac{2V_{,\varphi}}{\kappa\varphi_0' V}\right)\mathcal{R}' = 0 \quad (88)$$

$$\begin{aligned}\tilde{\mathcal{R}}'' + Q^2\tilde{\mathcal{R}} - \left(3 - \frac{\kappa}{2}\varphi_0'^2\right)\left(1 + \frac{2V_{,\varphi}}{\kappa\varphi_0' V}\right)\tilde{\mathcal{R}}' + 2F_a Q^2\mathcal{R} \\ - \left(3 - \frac{\kappa}{2}\varphi_0'^2\right)\left[\left(\left(1 + \frac{\kappa}{6}\varphi_0'^2\right)\frac{V_{,\varphi}}{V} + \frac{2V_{,\varphi\varphi}}{\kappa\varphi_0' V}\right)\tilde{\varphi}_0\right. \\ \left. - \left(\frac{\kappa\varphi_0'}{3} + \frac{2V_{,\varphi}}{\kappa\varphi_0'^2 V}\right)\tilde{\varphi}_0' + \left(3 + \frac{\kappa}{2}\varphi_0'^2 + \frac{6V_{,\varphi}}{\kappa\varphi_0' V}\right)F_a\right]\mathcal{R}' = 0, \quad (89)\end{aligned}$$

with:

$$Q = \frac{k}{Ha}.$$

Conclusions:

We have proposed a modified gravity model with good properties at the quantum level. It is finite on shell in the vacuum and only lives at one loop.

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We obtain that a photon, or a massless particle, moves in a null geodesic of $\mathbf{g}_{\mu\nu} = g_{\mu\nu} + \kappa_2 \tilde{g}_{\mu\nu}$, but the proper time is given by $g_{\mu\nu}$. On the other side, a massive particle do not move in a standard geodesic, but it is a second order equation.

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We solved the Schwarzschild case outside matter. This solution could be used to study a black hole. For the sun, we used the Newtonian approximation and found a new deflection of light. $\tilde{\delta}$ matter is $< 1\%$ of the total mass at a solar system scale. This means that the modification of $\tilde{\delta}$ gravity is not important at a solar system scale. However, a different result could be find at other scales.

Conclusions:

We studied the Non-Relativistic case. In the Newtonian limit, we obtained a similar expression as in GR, where we have an effective potential. This potential depends on $\rho^{(0)}$ and $\tilde{\rho}^{(0)}$, where the last one correspond to $\tilde{\delta}$ matter. We found a relation between $\rho^{(0)}$ and $\tilde{\rho}^{(0)}$.

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We used this relation to study a few density profiles: Spherically Homogeneous, Exponential, Einasto and Navarro-Frenk-White profiles. An amplified effect in the rotation velocity is produced by $\tilde{\delta}$ matter.

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We used this relation to study a few density profiles: Spherically Homogeneous, Exponential, Einasto and Navarro-Frenk-White profiles. An amplified effect in the rotation velocity is produced by $\tilde{\delta}$ matter.

We find an exact solution for the cosmological case and obtain that $\tilde{\delta}$ gravity do not require dark energy. With this exact solution, we could also study very early phenomenon in the universe.

Conclusions:

In this model, the universe only has non relativistic matter and radiation. A $1 \gg C \neq 0$ is necessary to obtain an accelerated expansion of the universe. This model ends in a Big Rip.

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Finally, we introduce the cosmological inflation with $\tilde{\delta}$ gravity. The background equation have been introduced to explain the accelerated expansion such as dark energy.

Future Works:

- To explain the anomalous secular increase of the eccentricity of the orbit of the Moon, using the Newtonian and Post-Newtonian approximation (See L. Iorio. Mon.Not.Roy.Astron.Soc.415:1266-1275 (2011)).
- To use the exact Schwarzschild solution to study a black hole.
- To develop the $\tilde{\delta}$ standard model.
- To continue the study of cosmic inflation with $\tilde{\delta}$ gravity.
- To fit the CMB power spectrum with $\tilde{\delta}$ gravity.
- To study quantum effects at times close to the Big-Rip.
- etc...

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