Group Invariant Tranformations in Cosmology

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Scalar field cosmology

We consider the action

$$S = S_{EH} + S_\phi + S_m$$

where $S_{EH} = \int dx^4 \sqrt{-g} R$ is the Einstein-Hilbert action, R is the Ricci scalar of the underlying space, S_ϕ is the action of the scalar field

$$S_\phi = \int dx^4 \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} + V(\phi) \right]$$

and $S_m = \int dx^4 \sqrt{-g} L_m$ is the matter term. We assume that ϕ inherits the symmetries of underlying space, hence for the FLRW spacetime $\phi = \phi(t)$ and consequently $\phi_{;\nu} = \dot{\phi} \delta^0_\nu$ where $\dot{\phi} = \frac{d\phi}{dt}$.

Scalar field cosmology

For comoving observers ($u^\mu = \delta^\mu_0$) and for a FLRW spacetime, the Einstein field equations are

$$H^2 = \frac{\kappa}{3}\left(\rho_m + \rho_\phi\right) - \frac{K}{a^2}$$

$$3H^2 + 2\dot{H} = -\kappa(P_m + P_\phi) - \frac{K}{a^2}$$

where $H(t) \equiv \dot{a}/a$ is the Hubble function.
Furthermore, assuming that the scalar field and matter do not interact, we have the two following equations

$$\dot{\rho}_m + 3H(\rho_m + P_m) = 0$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

while the corresponding equation of state (EoS) parameters are given by $w_m = P_m/\rho_m$ and $w_\phi = P_\phi/\rho_\phi$, where

$$\rho_\phi \equiv \frac{1}{2}\dot{\phi}^2 + V(\phi)\ ,\ \ P_\phi \equiv \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

Minisuperspace

The Lagrangian of the field equations in a minimally coupled scalar field cosmology in a spatially flat FLRW spacetime with a perfect fluid with a constant equation of state parameter $P_m = (\gamma - 1)\,\rho_m$ is

$$L\left(a, \dot{a}, \phi, \dot{\phi}\right) = -3a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 - a^3 V\left(\phi\right) - \rho_{m0}\, a^{-3(\gamma-1)}. \tag{1}$$

where $\gamma = 1$ for dust, $\gamma = \frac{4}{3}$ for radiation. The field equations are the Euler-Lagrange equations of (1) with respect to the variables (a, ϕ) and the Hamiltonian conservation law, which in terms of the momenta $p_a = \frac{\partial L}{\partial \dot{a}}$, $p_\phi = \frac{\partial L}{\partial \dot{\phi}}$, becomes

$$-\frac{1}{12a}p_a^2 + \frac{1}{2a^3}p_\phi^2 + a^3 V\left(\phi\right) + \rho_{m0}\, a^{-3(\gamma-1)} = 0.$$

WdW Equation

The Wheeler-DeWitt equation is a Klein Gordon equation which is defined by the conformal Laplacian operator, i.e.,

$$\Delta\Psi + \frac{n-2}{4(n-1)}\bar{R}\left(x^k\right)\Psi + V_{eff}\left(x^k\right)\Psi = 0,$$

where Δ is the Laplacian operator of the minisuperspace $d\hat{s}^2 = -6ada^2 + a^3 d\phi^2$ and it is

$$\Delta \equiv -\frac{1}{6a}\left(\frac{\partial^2}{\partial a^2} + \frac{\partial}{\partial a}\right) + \frac{1}{a^3}\frac{\partial^2}{\partial \phi^2}. \tag{2}$$

Moreover, $V_{eff}\left(a,\phi\right) = 2a^3\left[V\left(\phi\right) + \rho_{m0}a^{-3\gamma}\right]$ and $n = 2$.

The potential

- Power law $V(\phi) = V_0 \phi^{-m}$
- Exponential: $V(\phi) = V_0 \exp(-\lambda\phi)$
- Unified dark energy: $V(\phi) = V_0 \left(1 + \cosh^2(\lambda\phi)\right)$, and many others.

GR is a geometric theory. Symmetries of differential equations can be used as a geometric selection rule for the unknown form of the potential

Geometric selection rules

- The Noether point symmetries are related with the Homothetic algebra of the minisuperspace.
  Results in scalar field cosmology: Exponential potential & Unified dark energy potential.
- We propose to use as a new selection rule the existence of Lie point symmetries in the WdW equation.

Hartle's Criterion

If the wavefunction of the universe is strongly peaked (it admits oscillatory terms) then we have correlations among the geometrical and matter degrees of freedom and classical trajectories are expected. (J. B. Hartle, Gravitation in Astrophysics, Cargese 1986, ed. by B. Carter & J.B Hartle)

Symmetries

Strongly peaked wavefunction indicates oscillatory terms. Oscillatory terms in the solution of a "Klein-Gordon" equation means "separation of variables" which means the existence of a group of point transformations which leaves invariant the WdW equation.

Invariant Functions

Let $F(x, y)$ be a function in M. Under a one-parameter point transformation the function becomes $\bar{F}(\bar{x}, \bar{y})$.

Definition (Invariant Functions)

The function F is invariant under the one-parameter point transformation if and only if $\bar{F}(\bar{x}, \bar{y}) = 0$ when $F(x, y) = 0$ at all points where the one parameter point transformation acts. Equivalently, the generator X of the point transformation is a symmetry of the function F if $X(F) = 0$.

The characteristic function or zeroth-order invariant W, of X, is defined as follows

$$dW = \frac{dx}{\xi(x, y)} - \frac{dy}{\eta(x, y)}.$$

Therefore any function of the form $F = F(W)$, where W is the zeroth-order invariant of X, is invariant under the one-parameter point transformation with generator X.

Lie Point symmetries

Consider the infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon \xi^i \left(x^k , u^B \right) \ , \ \bar{u}^A = u^A + \varepsilon \eta^A \left(x^k , u^B \right)$$

with generator $X = \frac{\partial \bar{x}^i}{\partial \varepsilon} \partial_i + \frac{\partial \bar{u}^A}{\partial \varepsilon} \partial_A$. Let $X^{[n]}$ be the nth-prolongation vector

$$X^{[n]} = X + \eta^A_{[i]} \partial_{u_{,i}} + ... + \eta^A_{[ij...i_n]} \partial_{u_{ij...i_n}}$$

Then the DE, $H \left(x^i , u^A , u^A_{,i} , u^A_{,ij} , .. \right)$, is invariant under the action of X if there exist a function λ, where the following condition holds

$$X^{[n]} \left(H \right) = \lambda H \quad , \quad \mathrm{mod}\, H = 0.$$

The infinitesimal generator X is called a Lie point symmetry of the PDE.

Noether Point Symmetries

In case of DEs H, which arise from a variational principle the following theorem holds.

Theorem (Noether's Theorem)

The generator X of a one-parameter point transformation of the Action Integral of a Lagrangian $L_P = L_P\left(x^k, u, u_k\right)$, which transforms the Action Integral in a way that the Euler-Lagrange equations are invariant is called a Noether symmetry.

Mathematically, X is a Noether symmetry if there exists a vector field $F^i = F^i\left(x^i, u\right)$ such that

$$X^{[1]} L_P + L_P D_i \xi^i = D_i F^i .$$

The generator X is called a Noether symmetry. The corresponding Noether flow is

$$\Phi^i = \xi^k \left(u_k \frac{\partial L}{\partial u_i} - L \right) - \eta \frac{\partial L}{\partial u_i} + F^i$$

and satisfies the condition $D_i \Phi^i$. Noether symmetries form a algebra which is a sublagebra of the Lie symmetries $(NS \subseteq LS)$.

Lie symmetries of the Wheeler-DeWitt

Let $X=\partial_J+\left(\frac{2-n}{2}\psi\Psi+a_0\Psi\right)\partial_\Psi$ be a Lie symmetry for the WDW equation. The zero order invariants of the vector field X follow from the Lagrange system

$$\frac{d\bar{x}^b}{0}=\frac{d\bar{x}^J}{1}=\frac{d\Psi}{\left(\frac{2-n}{2}\psi+a_0\right)\Psi}$$

which turn out to be

$$\Psi\left(\bar{x}^b,\bar{x}^J\right)=\Phi\left(\bar{x}^b\right)\exp\left(\int\left(\frac{2-n}{2}\psi+a_0\right)d\bar{x}^J\right). \qquad (3)$$

Equivalently, from the Lie Bäcklund symmetry $\bar{X}=\left(\Psi_J-\left(\frac{2-n}{2}\psi+a_0\right)\Psi\right)\partial_\Psi$ we obtain the following equation $\Psi_J-\left(\frac{2-n}{2}\psi+a_1\right)\Psi=a_2\Psi$ which gives (3).
The latter is an operator.

Symmetries as a Geometric selection rule

By selecting the form of the potential/theory with the requirement of the existence of conservation laws, it is equivalent with that we let the theory (the minisuperspace) to select the corresponding model (the potential) since symmetries are generated by the CKVs of the minisuperspace.

Corollary

For every Lie point symmetry of the WDW equation a Noetherian conservation law corresponds for the classical field equations.

The “frequency” of the oscillator term in the solution of the WDW is related with the value of the Noetherian conservation law.

The Model

A **special solution** for a spatially flat FLRW spacetime ($K = 0$) which contains a perfect fluid with a constant equation of state parameter $P_m = (\gamma - 1)\,\rho_m$ and a scalar field with a constant equation of state parameter $w_\phi = \gamma_\phi - 1 = P_\phi / \rho_\phi$, exist when (C. Rubano & J. D. Barrow, Phys. Rev. D. **64**, 127301 (2001))

$$V(\phi) = V\left(H_{0,}\rho_{m0}, \gamma_\phi\right)\left[\sinh\left(\sqrt{3}\frac{\gamma - \gamma_\phi}{\sqrt{\gamma_\phi}}(\phi - \phi_0)\right)\right]^{-\frac{2\gamma_\phi}{\gamma - \gamma_\phi}}. \quad (4)$$

We can see that the Lagrangian of the field equations does not admit any conservation law.
Consider now the potential of the form (L. A. Urena-Lopez, T. Matos, Phys. Rev. D **62,** 081302 (2000))

$$V(\phi) = \left[\alpha \cosh(p\phi) + \beta \sinh(p\phi)\right]^q \quad (5)$$

in which for small values of ϕ is a power law model, whereas for large ϕ is a exponential model.

Lie symmetries

The WdW equation admits (except the trivial symmetries) the Lie symmetries

$$X_1 = a^{\frac{3\mu}{2}}\left[\frac{\sqrt{6}}{6}a\sinh\left(\frac{\sqrt{6}}{4}\mu\phi\right)\partial_a + \cosh\left(\frac{\sqrt{6}}{4}\mu\phi\right)\partial_\phi\right] \tag{6}$$

$$X_2 = a^{\frac{3\mu}{2}}\left[\frac{\sqrt{6}}{6}a\cosh\left(\frac{\sqrt{6}}{4}\mu\phi\right)\partial_a + \sinh\left(\frac{\sqrt{6}}{4}\mu\phi\right)\partial_\phi\right]. \tag{7}$$

where the constants p, q, γ are:

$$p = \frac{\sqrt{6}}{4}\mu\ ,\ \ q = -\frac{4}{\mu} - 2\ ,\ \ \gamma = \mu + 2$$

.

Therefore, for $\mu = -1$, we have that $q = 2$, $\gamma = 1$ i.e. we have the UDM potential with dust.

Invariant Solution

We select $\alpha = 1,\ \beta = 0$ (the case $\alpha = 0,\ \beta = 1$ is equivalent to that case) and we apply the coordinate transformation (normal coordiantes)

$$a = \left(x^2 - y^2\right)^{-\frac{1}{3\mu}} \ , \ \ \phi = \frac{2\sqrt{6}}{3\mu} \operatorname{arctan} h\left(\frac{y}{x}\right). \tag{8}$$

The WdW equation becomes

$$\left(x^2 - y^2\right)^{\frac{1}{\mu}+1}\left[\Psi_{,yy} - \Psi_{,xx} + \left(2V_0' x^{-\frac{4}{\mu}-2} + 2\rho_{m0}\right)\Psi\right] = 0 \tag{9}$$

where $V_0' = \frac{3}{8}\mu^2 V_0,\ \rho_{m0}' = \frac{3}{8}\mu^2 \rho_{m0}$. The invariant solution is $\Psi\left(x, y\right) = e^{\alpha_0 y}\Phi\left(x\right)$ where

$$\Phi_{,xx} - \left(2V_0' x^{-\frac{4}{\mu}-2} + 2\rho_{m0}' + a_0^2\right)\Phi = 0. \tag{10}$$

Classical Solution

In terms of the coordinates (x, y) the solution of the null Hamilton-Jacobi equation is

$$S(x, y) = c_1 y \pm \int \sqrt{c_1^2 + 2\rho'_{m0} + 2V'_0 x^{-\frac{4}{\mu}-2}}. \tag{11}$$

Hence the reduced field equations are.

$$\left(x^2 - y^2\right)^{-\left(\frac{1}{\mu}+1\right)} \dot{x} = \mp \left(\frac{\partial S}{\partial x}\right) \;,\; \left(x^2 - y^2\right)^{-\left(\frac{1}{\mu}+1\right)} \dot{y} = c_1 . \tag{12}$$

We apply the transformation $d\tau = \left(x^2 - y^2\right)^{\frac{1}{\mu}+1} dt = a^{-3(\mu+1)} dt$, and the dynamical system becomes

$$x' = \mp \sqrt{c_1^2 + 2\rho'_{m0} + 2V'_0 x^{-\frac{4}{\mu}-2}} \;,\; y' = c_1 \tag{13}$$

Classical Solution

Special case c1=0

$$\begin{aligned}\frac{H^2(a)}{H_0^2} &= \Omega_{m0} a^{-3(\mu+2)} + \Omega_{\Lambda 0} a^{-6}\left(y_0^2\left[\frac{\Omega_{m0}}{\Omega_{\Lambda 0}} + \left(y_0^2 + a^{-3\mu}\right)^{-\frac{2}{\mu}-1}\right]\right) + \\ &\quad + \Omega_{\Lambda 0} a^{-6}\left(+a^{-3\mu}\left(y_0^2 + a^{-3\mu}\right)^{-\frac{2}{\mu}-1}\right)\end{aligned}$$

where in the limit $y_0^2 + a^{-3\mu} \approx a^{-3\mu}$,

$$\frac{H^2(a)}{H_0^2} = E^2(a) = \Omega_{m0} a^{-3(\mu+2)} + \Omega_{\Lambda 0}\left(1 + y_0^2\left[\frac{\Omega_{m0}}{\Omega_{\Lambda 0}} a^{-6} + a^{3\mu}\right]\right).$$

Classical Solution

Special case c1=0.

For $\gamma = 1$, dust fluid,

$$\frac{H^2(a)}{H_0^2} = \Omega_{m0}\,a^{-3} + \Omega_{\Lambda 0}\left[1 + 2y_0^2\,a^{-3} + y_0^2\left(\frac{\Omega_{m0}}{\Omega_{\Lambda 0}} + y_0^2\right)a^{-6}\right]. \quad (14)$$

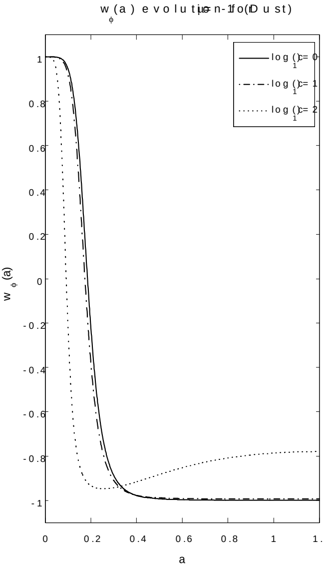
For $\gamma = \frac{4}{3}$, radiation fluid

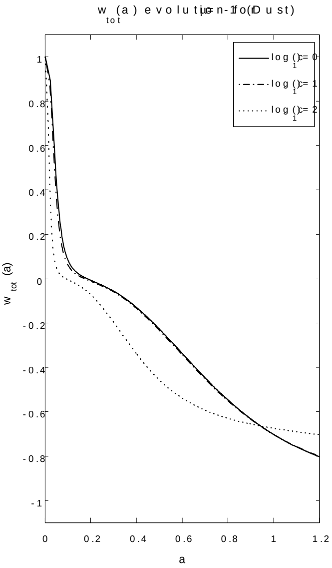
$$\frac{H^2(a)}{H_0^2} = \Omega_{r0}\,a^{-4} + \Omega_{\Lambda 0}\left[1 + 3y_0^2\,a^{-2} + 3y_0^4\,a^{-4} + y_0^2\left(\frac{\Omega_{r0}}{\Omega_{\Lambda 0}} + y_0^2\right)a^{-6}\right]. \quad (15)$$

For $\gamma = \frac{2}{3}$, curvature-like fluid

$$\frac{H^2(a)}{H_0^2} = \Omega_{K0}\,a^{-2} + \Omega_{\Lambda 0}\left[\sqrt{y_0^2 + a^4}\,a^{-2} + y_0^2\left(\frac{\Omega_{K0}}{\Omega_{\Lambda 0}} + \sqrt{y_0^2 + a^4}\right)a^{-6}\right]. \quad (16)$$

Evolution of the cosmological parameters





We introduce the dimensionless variables

$$x=\frac{\dot{\phi}}{\sqrt{6}H}\ ,\ y=\frac{\sqrt{V}}{\sqrt{3}H}\ ,\ \Omega_m=\frac{\rho_m}{3H^2}\ ,\ \lambda=-\frac{V_{,\phi}}{V} \tag{17}$$

and the lapse time $N=\ln a$. In the new variables the field equations reduce to the following first-order ODEs

$$\frac{dx}{dN}=-3x+\frac{\sqrt{6}}{2}\lambda y^2+\frac{3}{2}x\left[\left(1-w_m\right)x^2+\left(1+w_m\right)\left(1-y^2\right)\right] \tag{18}$$

$$\frac{dy}{dN}=-\frac{\sqrt{6}}{2}\lambda xy+\frac{3}{2}y\left[\left(1-w_m\right)x^2+\left(1+w_m\right)\left(1-y^2\right)\right] \tag{19}$$

$$\frac{d\lambda}{dN}=-\sqrt{6}\lambda^2\left(\Gamma-1\right)x \tag{20}$$

where $x^2+y^2\leq 1$.

Fixed points

For $\beta = 0$, and $\Gamma\left(\lambda\right) = 1 + \frac{qp^2}{\lambda^2} - \frac{1}{q}$, we find the following fixed points

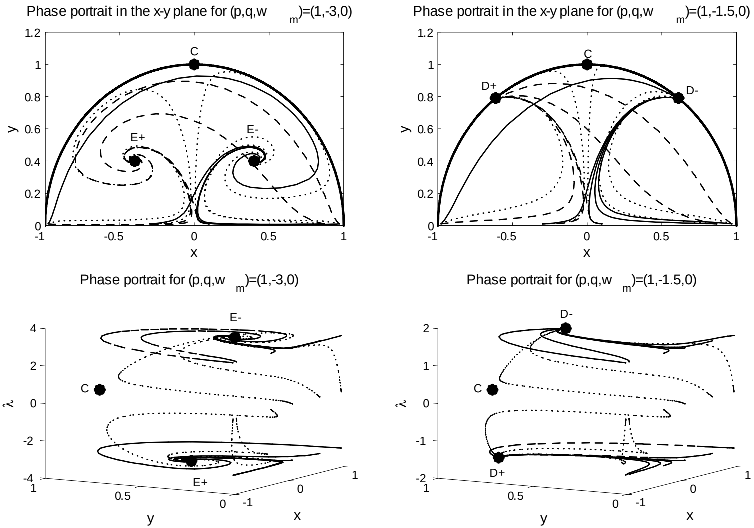
Point	$(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda})$	Ω_m	$\mathbf{w}_\phi$
O	$(0, 0, \lambda)$	1	$\nexists$
$A_{(\pm)}$	$(1, 0, \pm qp)$	0	1
$B_{(\pm)}$	$(-1, 0, \pm qp)$	0	1
C	$(0, 1, 0)$	0	-1
$D_{(\pm)}$	$\left(\pm\frac{\sqrt{6}}{6}qp, \sqrt{1-\frac{(qp)^2}{6}}, \pm qp\right)$	0	$-1 + \frac{(qp)^2}{3}$
$E_{(\pm +)}$	$\left(\pm\frac{\sqrt{6}(1+w_m)}{2qp}, \frac{\sqrt{6}\sqrt{1-w_m^2}}{2qp}, \pm qp\right)$	$1 - \frac{3(1+w_m)}{(qp)^2}$	w_m
$E_{(\pm -)}$	$\left(\pm\frac{\sqrt{6}(1+w_m)}{2qp}, -\frac{\sqrt{6}\sqrt{1-w_m^2}}{2qp}, \pm qp\right)$	$1 - \frac{3(1+w_m)}{(qp)^2}$	w_m

Stability

Table: Fixed points and their stability for the general potential and for the integrable subcases

Point	**Stability**	$\mu \in (-2, 0)$
O	Unstable	Unstable
$A_{(\pm)}$	Unstable	Unstable
$B_{(\pm)}$	Unstable	Unstable
C	Stable, $q \in \mathbb{R}^{*+}$	**Stable**
$D_{(\pm)}$	Stable, $q \in \mathbb{R}^{*-}$	Unstable
$E_{(\pm,+)}$	Stable, $q \in \mathbb{R}^{*-}$, $p < 0$	$\nexists$
$E_{(\pm,-)}$	Stable, $q \in \mathbb{R}^{*-}$, $p > 0$	$\nexists$

Fixed Points



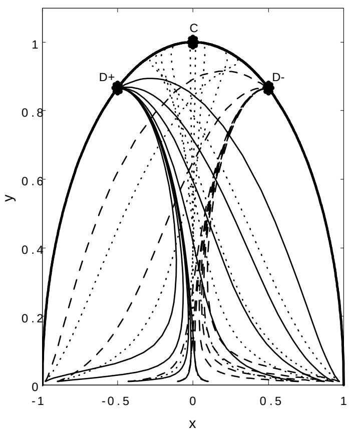
Phase portrait in the x-y plane for $(p,q,w_m)=(1,-3,0)$

Phase portrait in the x-y plane for $(p,q,w_m)=(1,-1.5,0)$

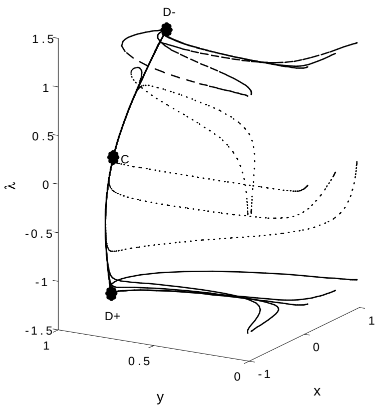
Phase portrait for $(p,q,w_m)=(1,-3,0)$

Phase portrait for $(p,q,w_m)=(1,-1.5,0)$

Fixed Points



Phase portrait in the x-y plane for μ=1 (Dust)



Phase portrait for μ=1 (Dust)

Comparison with cosmological data

We perform a joint likelihood analysis using the SNIa, BAO and the $H\left(z\right)$ data. The likelihood function is

$$\mathcal{L}\left(\mathbf{p}\right) = \mathcal{L}_{SNIa} \times \mathcal{L}_{BAO} \times \mathcal{L}_{H(z)} \tag{21}$$

where $\mathbf{p}$ is the statistical vector that contains the free parameters and $\mathcal{L}_A \propto e^{-\chi_A^2/2}$; that is, $\chi^2 = \chi^2_{SNIa} + \chi^2_{BAO} + \chi^2_{H(z)}$.

Comparison with cosmological data

ΛCDM (1df)	Ω_{m0}	w_Λ (fixed)	χ^2_{min}
SNIa+BAO	0.29	-1.000	560.32
SNIa+BAO+H(z)	0.29	-1.000	574.77

UCDM (2df)	Ω_{m0}	$w_{\phi 0}$	χ^2_{min}
Quint.			
SNIa+BAO	0.25	-0.965	563.48
SNIa+BAO+H(z)	0.28	-0.968	577.82
Phantom			
SNIa+BAO	0.29	-1.017	562.93
SNIa+BAO+H(z)	0.30	-1.016	576.59

With the use of Akaike information criterion we can say that these models are statistically equivalence (H. Akaike, IEEE Transactions of Automatic Control, **19**, 716 (1974)).

Brans-Dicke field

The Lagrangian of the field equations of a BD cosmological model with matter source is

$$\mathcal{L}\left(a,\dot{a},\phi,\dot{\phi}\right)=-3a\phi\dot{a}^{2}-3a^{2}\dot{a}\dot{\phi}+\frac{1}{2}\frac{\omega_{BD}}{\phi}a^{3}\dot{\phi}^{2}-a^{3}V\left(\phi\right)-\rho_{m0}a^{-3w}.$$

Where the minisuperspace is

$$ds_{(\gamma)}^{2}=-6a\phi da^{2}-6a^{2}dad\phi+\frac{\omega_{BD}}{\phi}a^{3}d\phi^{2}. \tag{22}$$

The WdW equation admit Lie symmetries when $V\left(\phi\right)=V_{0}\phi^{\lambda}$, for $\lambda_{1}=\left(1+w\right)\left(1-w\right)^{-1}$, and, (b) $2\lambda_{2}=\left(\sqrt{6\omega_{BD}+9}-3\right)\left(w+1\right)$

Brans-Dicke field WDW

Let us consider the case λ_1, for $w \neq 1$. Then we have that the invariant solution of the WdW equation is $\Psi\left(x,y\right)=\sum_{\beta} e^{-\beta x}\Phi\left(y\right)$ where

$$y m_2 \Phi_{,yy} + \left(2\beta m_1 + m_2\right)\Phi_{,y} + 2\left(\bar{V}_0 y^{\frac{1+w}{1-w}} + \bar{\rho}_{m0} + \frac{\omega_{BD}}{3y}\beta^2\right)\Phi = 0.$$

and

$$a=\exp\left(x\right)\ ,\ \phi = y\exp\left[3\left(w-1\right)x\right]$$

Brans-Dicke field Classical Solution

The classical solution follows from the solution of the following system

$$e^{3wx}\dot{x} = m_1 \left(\frac{\partial S}{\partial y}\right) - \frac{\omega_{BD}}{3y}\left(\frac{\partial S}{\partial x}\right), \tag{23}$$

$$e^{3wx}\dot{y} = m_1 \left(\frac{\partial S}{\partial x}\right) - m_2 y \left(\frac{\partial S}{\partial y}\right), \tag{24}$$

where from $p_x = 0$, which corresponds to $I_0 = 0$, the solution becomes

$$H(a)^2 = H_0^2 \left(\Omega_{\phi 0} a^{q_1(\omega_{BD}, w)} + \Omega_{m0} a^{q_2(\omega_{BD}, w)}\right), \tag{25}$$

Brans-Dicke field Special Solution

Case (A): Cosmological constant with dust. This means that $(q_1, q_2) = (0, -3)$ or $(q_1, q_2) = (-3, 0)$, from which we have $(w, \omega_{BD}) = \left(0, \frac{1}{6}\right)$ or $(w, \omega_{BD}) \simeq (0.28, -0.77)$.

Case (B): Dust with radiation. This requires, $(q_1, q_2) = (-3, -4)$ or $(q_1, q_2) = (-4, -3)$, hence $(w, \omega_{BD}) \simeq (0.63, 0)$ or $(w, \omega_{BD}) \simeq (0.55, -1)$.

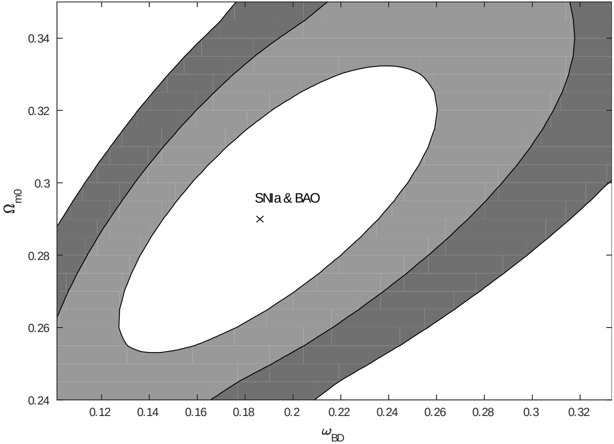
Case (C): Cosmological constant with radiation fluid This requires $(q_1, q_2) = (0, -4)$ or $(q_1, q_2) = (-4, 0)$, hence $(w, \omega_{BD}) = (0, 0)$, $(w, \omega_{BD}) = \left(\frac{1}{3}, 0\right)$ or, $(w, \omega_{BD}) = \left(\frac{1}{3}, -\frac{3}{4}\right)$.

Case (D): In the case of $(q_1, q_2) = (0, 0)$, which implies $(w, \omega_{BD}) = \left(-1, \frac{1}{6}\right)$ or $(w, \omega_{BD}) = \left(0, -\frac{4}{3}\right)$, from (25) we have a de Sitter solution.

It is interesting that when we assume a radiation fluid in the solution we have a solution in which $\omega_{BD} = 0$; however, this result is expected since when $\omega_{BD} = 0$, the action reduces to O'Hanlon's massive dilaton gravity, and consequently to $f(R)$-gravity in the metric formalism, which provides a radiation term.

Brans-Dicke field C.C.

Assuming that $w = 0$, i.e. dust then we have $H\left(a\right)^2 = H_0^2\left[\left(1-\Omega_{m0}\right)+\Omega_{m0}a^{-\frac{3\omega_{BD}+4}{3\omega_{BD}+1}}\right]$, hence from SNIa+BAO data we find that: $\Omega_{m0} = 0.29^{+0.032}_{-0.025}$, $\omega_{BD} = 0.19^{+0.075}_{-0.059}$, $\min(\chi^2) \simeq 564.29$.



What else can we learn from point symmetries?

Let us consider a simple well-known system in $\mathcal{M}^2$ space which follows from the Lagrangian

$$L\left(x,\dot{x},y,\dot{y}\right)=\frac{\Omega}{2}\left(\dot{x}^2-\dot{y}^2\right)-\frac{\mu}{2}\left(x^2-y^2\right), \tag{26}$$

where Ω,μ are constants. That Lagrangian describes the two-dimensional "oscillator" in a flat space with Lorentzian signature. It is straightforward to see that admits eight Noether point symmetries. Also the dynamical system in which Lagrangian (26) describes is invariant under the following discrete transformations $\{x\rightarrow -x\}$, or $\{y\rightarrow -y\}$, and the complex one $\{x\rightarrow iy\ ,\ y\rightarrow ix\}$, while, under the transformation $\{x\rightarrow y,\ y\rightarrow x\}$, the new Lagrangian $\bar{L}$, is $\bar{L}=-L$. However while the field equations have same the sign, the Hamiltonian constant changes: if the latter is zero then everything is invariant. These discrete symmetries follows directly from the existence of the Noether symmetry vectors.

What else can we learn from point symmetries?

Consider now the coordinate transformation

$$u=\frac{\sqrt{2}}{2}\left(x+y\right)~,~v=\frac{\sqrt{2}}{2}\left(x-y\right). \tag{27}$$

Where now the discrete transformation $\left\{x\rightarrow -x,y\rightarrow -y\right\}$ in the new coordinates system corresponds to $\left\{u\rightarrow -u~,~v\rightarrow -v\right\}$, while the transformation $\left\{x\rightarrow -x\right\}$ corresponds to $\left\{u\rightarrow -v~,~v\rightarrow -u\right\}$, and $\left\{y\rightarrow -y\right\}$ corresponds to the discrete transformation $\left\{u\rightarrow v~,~v\rightarrow u\right\}$. All these transformations are related with the admitted translation group of the flat space.
In the new coordinate system $\left\{u,v\right\}$, Lagrangian becomes

$$L\left(u,\dot{u},v,\dot{v}\right)=\Omega\left(\dot{u}\dot{v}\right)-\mu\left(uv\right). \tag{28}$$

We assume the second coordinate transformation

$$a = u^{p_+} v^{p_-} \ , \ \phi = \ln\left(u^{q_+} v^{q_-}\right), \tag{29}$$

in which the constants $p_{\pm}(\omega)$, $q_{\pm}(\omega)$ where $\Omega = 8\frac{(\omega-6)}{3\omega-16}$.
In the new coordinates the Lagrangian becomes

$$L\left(a, \dot{a}, \phi, \dot{\phi}\right) = e^{-2\phi}\left(6a\dot{a}^2 - 12a^2\dot{a}\dot{\phi} + \omega a^3\dot{\phi}^2 - \Lambda a^3\right), \tag{30}$$

where the constant $\Lambda = \mu$. The latter Lagrangian has the form of Dilaton field with a difference in the coefficient of $\dot{\phi}^2$. Furthermore, it is easy to see that, under the transformation $\phi = -\frac{1}{2}\ln(\psi)$, Lagrangian (30) becomes of Brans-Dicke form where $\omega = 4\omega_{BD}$.

From transformations (29), we see that the discrete transformation $\{y \rightarrow -y\}$, in the Cartesian coordinates, or $\{u \rightarrow v\ ,\ v \rightarrow u\}$ in the coordinates $\{u, v\}$, is that of the scale-factor duality (M. Gasperini and G. Veneziano, Phys. Rept. **373** (2003) 1)

$$a \rightarrow a^{-1} \tag{31}$$

if and only if $p_+ = -p_-$. From where we find, we find the unique solution $\omega = 4$, or $\omega_{BD} = 1$, where (30) corresponds to the Lagrangian of the dilaton field. Therefore we conclude that the scale-factor duality transformation is related to the existence of Noether symmetries of the Lagrangian and the field equations. The reason why only the constant $\omega_{BD} = 1$ is admitted for the a scale-factor duality follows from the property of the coordinate transformation (29).

In general for $\omega_{BD} \neq 1$, there exists the following discrete symmetry which is related with the scale-factor duality of the dilaton field

$$a \longrightarrow a^{(p_- Q_- - p_+ Q_+)} \exp\left(\left(p_+ P_+ - p_- P_-\right)\phi\right), \tag{32}$$

$$\exp(\phi) \longrightarrow a^{(q_- Q_- - q_+ Q_+)} \exp\left(\left(q_+ P_+ - q_- P_-\right)\phi\right), \tag{33}$$

or in the coordinates of the BD field

$$a \longrightarrow a^{(p_- Q_- - p_+ Q_+)} \psi^{-\frac{1}{2}(p_+ P_+ - p_- P_-)}, \tag{34}$$

$$\psi \longrightarrow a^{-2(q_- Q_- - q_+ Q_+)} \psi^{(q_+ P_+ - q_- P_-)}. \tag{35}$$

(A. Paliathanasis & S. Capozziello, arXiv:1602.08914)

Conclusions

- By using the Lie symmetries of the WdW equation we can construct quantum operators
- Construct classical conservation laws and find new integrable systems.
- Symmetries can provide us with cosmological viable models (see also S. Basilakos and J.D. Barrow PRD 91, 103517, (2015))
- From the existence of symmetries new discrete transformations/symmetries can be found.