Degrees of freedom in modified teleparallel gravity

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Outline

TEGR and f(T) gravity

Introduction to teleparallel gravity

The main object in teleparallel gravity is the tetrad field (or vierbein).

$$\underbrace{g_{\mu\nu}(x)}_{\text{General Relativity}} \longrightarrow \underbrace{e^a{}_\mu(x)}_{\text{Teleparallel Gravity}}$$

$$\underbrace{e_a = e_a^\mu \partial_\mu;}_{\text{vectors}} \quad \underbrace{e^a = e^a_\mu dx^\mu}_{\text{co-vectors}} \quad a = 0, 1, 2, 3.$$

$$\underbrace{\eta_{ab} = g_{\mu\nu} e^\mu{}_a e^\nu{}_b}_{\text{orthonormality condition}} \;, \quad \text{with} \quad \eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Introduction to teleparallel gravity

$$\boxed{e_a{}^{\mu}\ \underbrace{e^b{}_{\mu}}_{\text{co-frame}} = \delta^b_a \longrightarrow \underbrace{g_{\mu\nu}}_{\text{metric}} = \eta_{ab}\, e^a{}_{\mu}\, e^b{}_{\nu} \longleftrightarrow \sqrt{-g} = \det[e^a{}_{\mu}] = e.}$$

It is defined the Weitzenböck connection

$$\boxed{\overset{\mathrm{W}}{\Gamma}{}^{\mu}_{\ \rho\nu} = e^{\mu}_{a}\, \partial_{\nu} e^{a}_{\rho} = -e^{a}_{\rho}\, \partial_{\nu} e^{\mu}_{a} \ ,}$$

Covariant derivative

$$\boxed{\nabla_{\nu} V^{\mu} = \partial_{\nu}(V^{a} e^{\mu}_{a}) + \overset{\mathrm{W}}{\Gamma}{}^{\mu}{}_{\rho\nu} V^{\rho} = e^{\mu}_{a} \partial_{\nu} V^{a}}$$

Introduction to teleparallel gravity

- The manifold whose connection is $\overset{\mathrm{W}}{\Gamma}{}^{\mu}{}_{\rho\nu}$ defines the Weitzenböck spacetime. Its main characteristic is that it has absolute parallelism: $\nabla_\nu e_a^\mu = 0$.
- The torsion tensor (torsion of the Weitzenböck connection $\overset{\mathrm{W}}{\Gamma}{}^{\mu}_{\rho\nu}$)

$$\boxed{T^{\mu}{}_{\nu\rho} = e_a^{\mu}(\partial_\nu e^a{}_\rho - \partial_\rho e^a{}_\nu)}\,.$$

It resembles the electromagnetic field tensor $F_{\mu\nu}$.

- It is defined also the contorsion tensor

$$\boxed{K^{\mu}{}_{\rho\nu} = \overset{\mathrm{W}}{\Gamma}{}^{\mu}_{\rho\nu} - \Gamma^{\mu}_{\rho\nu}}$$

A. Einstein, Sitzungsber. Preuss. Akad. Wiss. (1928), 217; ibid., (1939), 401; A. Einstein, Math. Annal. 102 (1930), 685.

Equivalence between TG and GR

Torsion scalar is defined as

$$\boxed{\mathcal{T} = S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu},}$$

where the so-called superpotential is

$$2\,S_\rho{}^{\mu\nu} \equiv \underbrace{\frac{1}{2}\left(T_\rho{}^{\mu\nu} - T^{\mu\nu}{}_\rho + T^{\nu\mu}{}_\rho\right)}_{contorsion\ K^{\mu\nu}{}_\rho} + T_\lambda{}^{\lambda\mu}\,\delta^\nu_\rho - T_\lambda{}^{\lambda\nu}\,\delta^\mu_\rho\,.$$

There is an equivalence between Ricci and torsion scalar

$$\boxed{R = -T + \frac{2}{e}\partial_\rho(e T_\mu{}^{\mu\rho})}$$

which becomes an equivalence at the level of the actions, by integrating out the four-divergence.

Teleparallel gravity lagrangian

TEGR action

$$\boxed{S_{TEGR}[e^a] = \frac{1}{2\kappa}\int d^4x\; e\; S_\rho{}^{\mu\nu}\, T^\rho{}_{\mu\nu},}$$

S_{TEGR} quadratic in torsion tensor $\implies$ first order derivatives of the tetrad field. The equations of motion are

$$\boxed{\frac{1}{e}\partial_\mu(e\; S_a{}^{\mu\nu}) + e_a^\lambda(S_\rho{}^{\mu\nu}\, T^\rho{}_{\mu\lambda} - \frac{1}{4}\delta_\lambda^\rho S_\rho{}^{\mu\nu}\, T^\rho{}_{\mu\nu}) = 4\pi G\; e_a^\lambda T_\lambda^\rho}$$

where T_λ^ρ is the energy-momentum tensor

Geodesic equation in TEGR

- The action for a spinless particle with mass m in a gravitational field is

$$\boxed{S_{TEGR} = -m\int u_a e^a_\mu dx^\mu \longleftrightarrow S_{GR} = -m\int ds}$$

- Starting from S_{TEGR}, it is obtained an equation of motion where the torsion plays the role of the gravitational force

$$\boxed{\frac{du_\mu}{ds} - \overset{\mathrm{W}}{\Gamma}{}^{\lambda}{}_{\mu\nu} u_\lambda u^\nu = \overset{\mathrm{W}}{T}{}^{\lambda}{}_{\mu\nu} u_\lambda u^\nu}$$

- The trajectory of the particle is the same, nonetheless the interpretation of the gravitational action is conceptually different.

R. Aldrovandi and J. G. Pereira, "An introduction to Teleparallel Gravity" (2010)

GR and TEGR

Table: GR versus TEGR

• Gravity encoded in curvature via torsionless Levi-Civita connection	• Gravity encoded in torsion via curvatureless Weitzenböck connection
• $S_{GR} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\ R$	• $S_{TEGR} = \frac{1}{2\kappa} \int d^4x\ e\ T$
• R is the Ricci (curvature) scalar	• T is the Weitzenböck (torsion) scalar
• R has 2nd derivatives of the metric	• T has 1st derivatives of the tetrad
• The metric has 10 indep. comp.	• The tetrad has 16 indep. comp.
	but LLI of L_{TEGR} reduces it to 10.
• Particles moves along geodesics	• Particles follows a force equation

$f(T)$: modified teleparallel gravity

The idea:

$$\boxed{R \longrightarrow f(R) \quad T \longrightarrow f(T)} \quad \Longrightarrow \quad \boxed{S_T[e^a] = \frac{1}{2\kappa}\int d^4x \; e \; f(\mathcal{T})}$$

- Equations of motion in $f(\mathcal{T})$ theories are always second order, since $L_{f(T)}$ does not contain 2nd derivatives in the tetrad.

$$\boxed{\left(\frac{4}{e}\partial_\mu(eS_a{}^{\mu\nu}) + 4e_a^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\mu\nu}\right) f'(\mathcal{T}) + 4S_a{}^{\mu\nu}\partial_\mu\mathcal{T} f''(\mathcal{T}) - e_a^\nu f(\mathcal{T}) = -2\kappa e_a^\lambda T_\lambda^\nu,}$$

(T_λ^ν is the energy-momentum tensor).

R. Ferraro and F. Fiorini, Phys. Rev. D 75, 084031 (2007), R. Ferraro and F. Fiorini, Phys. Rev. D 78, 124019 (2008)

Local Lorentz Invariance

- These 2nd order equations of motion come together with the loss of LLI, since under a LLT in the tangent space

$$\boxed{e^b \rightarrow e^{b'} = \Lambda^{b'}_{\ a}(x) e^a,}$$

the transformed tetrad $e^{b'}$ is not a solution of the field equations. Nonetheless, this transformation keep the metric $g_{\mu\nu}$ unchanged.

- The four-divergence that distinguishes TEGR with GR is not invariant under this transformation. For a general f we have

$$\boxed{f(\mathcal{T}) = f\left(-R + \frac{2}{e}\partial_\mu (T^{\rho}{}_{\rho}{}^{\mu})\right).} \implies f(T) \text{ not LLI}$$

- This apparent loss of local Lorentz invariance can be seen as by the theory having more degrees of freedom.

f(T): smoothing cosmological and conical singularity

- Born-Infeld-modified teleparallelism solves the particle horizon problem in a spatially flat FLRW universe by providing an initial exponential expansion without resorting to an inflaton field.
- In Born-Infeld teleparallel gravity it is found an exact 3D vacuum circular symmetric solution without cosmological constant, consisting in rotating spacetime with non singular behaviour.
- The spacetime behaves at infinity as the conical geometry in 3D GR without cosmological constant. However, the solution has no conical singularity because the space ends at a minimal circle that no freely falling particle can ever reach in a finite proper time.

R. Ferraro and F. Fiorini, Phys. Rev. D 75 (2007) 084031

R. Ferraro and F. Fiorini, Phys.Lett. B692 (2010) 206-211

$f(T)$ with $T=0$

- If a vacuum solution of $f(T)$ gravity has $T=0$, then it will be a solution of TEGR as well (a cosmological constant might be necessary)
- Equations of motion become

$$\boxed{4e^{-1}\partial_\mu(ee_a^\lambda S_\lambda{}^{\mu\nu})+4e_a^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\mu\nu}-e_a^\nu\,\frac{f(0)}{f'(0)}=0,}$$

  which is a TEGR vacuum equation with cosmological constant $2\Lambda=f(0)/f'(0)$.
- The cosmological constant term can be avoided by restricting the family of functions f to those having $f(0)=0$, $f'(0)\neq 0$.
- We can exploit the freedom to do LLT in TEGR to look for a tetrad having $T=0$; if we success, then we will state that such solution survives in $f(T)$ gravity.

R. Ferraro and F. Fiorini, Phys. Rev. D **84**, 083518 (2011b)

f(T) in null tetrad approach

TEGR in null tetrad approach

$$\boxed{\underbrace{e^a(x)}_{\text{orthonormal}} \longrightarrow \underbrace{n^a{}_\mu(x)}_{\text{null tetrad}}}$$

$$\boxed{\{n^a\} = \{\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}\}}$$

$$\boxed{l = \frac{(e^0 + e^1)}{\sqrt{2}}, \quad n = \frac{(e^0 - e^1)}{\sqrt{2}} \quad m = \frac{(e^2 + ie^3)}{\sqrt{2}} \quad \overline{m} = \frac{(e^2 - ie^3)}{\sqrt{2}}}$$

C. Bejarano, R. Ferraro and M. J. Guzmán, Eur. Phys. J. C 75 (2015) 77

TEGR in null tetrad approach

This tetrad form a null basis

$$\boxed{\mathbf{l}\cdot\mathbf{l}=0,\ \ \mathbf{n}\cdot\mathbf{n}=0,\ \ \mathbf{m}\cdot\mathbf{m}=0,\ \ \overline{\mathbf{m}}\cdot\overline{\mathbf{m}}=0\ ,}$$

but it is not orthornormal

$$\boxed{\mathbf{l}\cdot\mathbf{n}=1,\ \ \mathbf{m}\cdot\overline{\mathbf{m}}=-1,\ \ \mathbf{l}\cdot\mathbf{m}=0,\ \ \mathbf{n}\cdot\mathbf{m}=0\ .}$$

We get the metric in terms of the null tetrad $g_{\mu\nu}\ =\ \eta_{ab}\, n^a_\mu\, n^b_\nu$, where η_{ab} is now

$$\boxed{\eta_{ab}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}.}$$

C. Bejarano, R. Ferraro and M. J. Guzmán, Eur. Phys. J. C 75 (2015) 77

The metric reads

$$\boxed{\mathbf{g} = \mathbf{n} \otimes \mathbf{l} + \mathbf{l} \otimes \mathbf{n} - \mathbf{m} \otimes \overline{\mathbf{m}} - \overline{\mathbf{m}} \otimes \mathbf{m}\ .}$$

We will use the fact that the following transformation leaves the metric unchanged

$$\boxed{\mathbf{l} \longrightarrow \ \exp[\lambda(\mathbf{x})]\ \mathbf{l}\ , \qquad\qquad \mathbf{n} \longrightarrow \ \exp[-\lambda(\mathbf{x})]\ \mathbf{n}}$$

but transforms the tetrad with a local Lorentz boost along the direction of $\mathbf{e}^1$ with parameter $\gamma(\mathbf{x}) = \cosh[\lambda(\mathbf{x})]$.

Null tetrad in Kerr geometry

The chosen null tetrad is

$$\mathbf{n}^a_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^\lambda \left(1 - \frac{2mr}{\Sigma}\right) & e^\lambda \left(1 + \frac{2mr}{\Sigma}\right) & 0 & e^\lambda \left(1 + \frac{2mr}{\Sigma}\right) a\, s^2(\theta) \\ e^{-\lambda} & -e^{-\lambda} & 0 & -e^{-\lambda}\, a\ s^2(\theta) \\ 0 & 0 & r + iac(\theta) & (r + iac(\theta)) is(\theta) \\ 0 & 0 & r - iac(\theta) & -(r - iac(\theta)) is(\theta) \end{pmatrix}$$

where $\lambda = \lambda(t, r, \theta)$. With this, the Weitzenböck invariant becomes

$$T = \frac{2}{\Sigma^3} \ \left(\Sigma^2 - \ 4\, a^2 \cos^2\theta \ (\Sigma + \ m\, r) - \ 2\, r\, \Sigma^2\ \partial_t \lambda\right) \ .$$

Null tetrad in Kerr geometry

There is a family of functions $\lambda(t,r,\theta)$ that realize the vanishing of T

$$\boxed{\lambda(t,r,\theta) \;=\; \frac{t}{2\,r}\,\left(1-\;4\,a^2\cos^2\theta\,\frac{\Sigma+\;m\,r}{\Sigma^2}\right)+\lambda_1(r,\theta)\;.}$$

Null tetrad in McVittie geometry

The McVittie metric

$$\boxed{ds^2 = -\left(\frac{1-\mu}{1+\mu}\right)^2 dt^2 + (1+\mu)^4 a^2(t) d\mathbf{x}^2}$$

with

$$\mu = \frac{m}{2a(t)|\mathbf{x}|}$$

is rewritten, following the coord. transformation $\mathbf{R} = (1+\mu)^2 a(t)\mathbf{x}$, as

$$\boxed{ds^2 = \left(1 - \frac{2m}{R} - H^2R^2\right) dt^2 + \frac{2RH}{\sqrt{1-\frac{2m}{R}}} dtdR - \frac{dR^2}{1-\frac{2m}{R}} - R^2\left[d\theta^2 + \sin(\theta)^2 d\phi^2\right]}$$

C. Bejarano, R. Ferraro and M. J. Guzmán, "McVittie geometry in f(T) gravity" [work in preparation]

Null tetrad in McVittie geometry

The choice of the null tetrad is

$$\boxed{{n^a}_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\lambda(t,R)}\left(\sqrt{1-\frac{2m}{R}} + RH(t)\right) & -e^{-\lambda(t,R)}\frac{1}{\sqrt{1-\frac{2m}{R}}} & 0 & 0 \\ e^{\lambda(t,R)}\left(\sqrt{1-\frac{2m}{R}} - RH(t)\right) & e^{\lambda(t,R)}\frac{1}{\sqrt{1-\frac{2m}{R}}} & 0 & 0 \\ 0 & 0 & R & iR\sin(\theta) \\ 0 & 0 & R & -iR\sin(\theta) \end{pmatrix}}$$

which gives an expression for the torsion scalar T as a function of λ

$$\boxed{T = \frac{2 - 6\ r^2\ H(t)^2\ -\ 4\ r\ \lambda^{(1,0)}(t,r)}{r^2}.}$$

It is possible to find a $\lambda(t,R)$ such that the torsion vanishes, this is

$$\boxed{\lambda(t,R) = \frac{t}{2R} - \frac{3R}{2}\int H(t)^2 dt.}$$

C. Bejarano, R. Ferraro and M. J. Guzmán, "McVittie geometry in f(T) gravity" [work in preparation]

Conformal transformations in f(T)

Equivalence with Brans-Dicke theory

- The study of the behaviour of the theory under conformal transformations could reveal the nature of these extra d.o.f..
- For this, it is useful to work with the Brans-Dicke equivalence. The action for $f(\mathcal{T})$ can be generalized as

$$\boxed{S = \frac{1}{2\kappa^2}\int d^4xe\left(f(\phi) + (T-\phi)f'(\phi)\right) + S_m(e^a_\mu)}$$

where it can be easily seen that $\phi = T$ if $f''(\phi) = 0$.

- Introducing a scalar field φ such that $F(\varphi) = f'(\phi)$ the action S can be written as

$$\boxed{S_{JF} = \frac{1}{2\kappa^2}\int d^4xe(F(\varphi)T - \omega(\varphi)g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - 2V(\varphi)) + S_m,}$$

where $2V(\varphi) = \phi f'(\phi) - f(\phi)$.

R.J. Yang, *Europhys. Lett.* 93, 60001 (2011)

Conformal transformation in $f(\mathcal{T})$

- $f(\mathcal{T})$ action in metric formalism corresponds to a Brans-Dicke theory with $\omega = 0$ in Jordan frame.
- $f(R)$ action is equivalent to Einstein-Hilbert action plus a scalar field action via a conformal transformation. We may think that $f(\mathcal{T})$ has the same feature.
- Consider a conformal transformation of the metric defined as $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, with $\Omega(x)$ a positive smooth, non-vanishing function.
- Under this transformation, the torsion transforms as

$$\boxed{\hat{T}^{\rho}_{\ \mu\nu} = T^{\rho}_{\ \mu\nu} + \Omega^{-1}(\delta^{\rho}_{\nu}\partial_{\mu}\Omega - \delta^{\rho}_{\mu}\partial_{\nu}\Omega).}$$

- With this, the teleparallel Lagrangian transforms as

$$\boxed{\hat{T} = \Omega^{-2}T + 4\Omega^{-3}\partial^{\mu}\Omega\, T^{\rho}_{\ \rho\mu} - 6\Omega^{-4}\partial_{\mu}\Omega\partial^{\mu}\Omega.}$$

R.J. Yang, *Europhys. Lett.* 93, 60001 (2011)

Conformal transformation in $f(\mathcal{T})$

- Using previous results, action for $f(\mathcal{T})$ can be written as

$$\begin{aligned} S_{EF} &= \frac{1}{2\kappa^2}\int d^4x\hat{e}\left(\hat{T}+2F^{-3}\hat{\partial}^{\mu}F\hat{T}^{\rho}_{\rho\mu}-\frac{1}{2}\hat{g}^{\mu\nu}\hat{\nabla}_{\mu}\psi\hat{\nabla}_{\nu}\psi-U(\psi)\right)\\ &\quad +S_m(F(\varphi)^{-1/2}\hat{e}^a_{\mu}) \end{aligned}$$

where $F(\varphi)=\Omega^2$, $U(\phi)=2V(\varphi)/F^2(\varphi)$, and $(d\phi/d\varphi)^2=2\omega/F-3[F'(\varphi)]^2/F^4$.

- Compared with the case of $f(R)$, there is an additional scalar-torsion coupling term, $2F^{-3}\hat{\partial}^{\mu}F\hat{T}^{\rho}_{\rho\mu}$ which cannot be removed by conformal transformations.
- In conclusion, $f(\mathcal{T})$ theories are not dynamically equivalent to TEGR action plus a scalar field action via a conformal transformation. Therefore, the hidden degrees of freedom are manifested in another, unknown mechanism.

Extra: Hamiltonian formulation of f(T)

- The number of physical degrees of freedom can be calculated through the analysis of the algebra of constraints in the Hamiltonian formulation of the theory.
- Through this procedure, Li et al. (2011) find 3 extra d.o.f. when compared with TEGR, but they do not state its physical interpretation. They suggest that they may be related to **one massive vector field** or **one massless vector field plus one scalar field**.
- However this work, and the form of this extra d.o.f. remains to be confirmed by further research.

R. Ferraro and M. J. Guzmán, Hamiltonian formalism for modified teleparallel gravity, in Gravitation, Relativistic Astrophysics and Cosmology, Proceedings of the Second Argentinian-Brazilian Meeting (Buenos Aires, 2014), AAA Workshop Series Volume 7 (2015), 99-104.

R. Ferraro and M. J. Guzmán, Hamiltonian formulation of general relativity in the teleparallel framework [work in preparation]

Conclusions

- TEGR is equivalent to GR up to a four-divergence in the Lagrangian, however equations of motion are equivalent.
- $f(T)$ loses LLI due to this term, and more degrees of freedom appear
- With the null-tetrad approach, we can parameterize one of these d.o.f. in a Lorentz boost along a direction.
- Through this method we find easily solutions with $T = 0$, like in Kerr and McVittie solutions.
- $f(T)$ is not equivalent to TEGR plus a scalar field action via a conformal transformation.