Holographic trace anomaly of GJMS operators

CosmoConce 2016

Fabrizzio Bugini

Work done with D.E. Diaz

8 de abril, 2016

Contents

Introduction

- Conformal symmetries are related to Weyl reescaling of metric tensor

$$S[g_{\mu\nu}, A] = S[g'_{\mu\nu}, A] \quad (1)$$

where

$$g'_{\mu\nu} = e^{2\Omega(x)} g_{\mu\nu} \quad (2)$$

- This symmetry implies for energy-momentum tensor defined by

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\partial S[g_{\mu\nu}, A]}{\partial g^{\mu\nu}}, \quad (3)$$

trace is equal to zero

$$T^{\mu}_{\mu} = 0 \quad (4)$$

Introduction

- After quantizing the corresponding theory, trace does not vanish

$$\langle T \rangle \neq 0 \Rightarrow \text{anomaly appears!} \tag{5}$$

D.M. Capper and M.J. Duff (1974)

- There are two kinds of anomaly coefficients related to Euler density (Type - A) and Weyl invariants (Type - B) (depending of dimension) where the Weyl tensor is defined by

$$\begin{aligned} W_{\mu\nu\alpha\beta} &= R_{\mu\nu\alpha\beta} - \frac{2}{d-2}\left(g_{\mu[\alpha}R_{\beta]\nu} - g_{\nu[\alpha}R_{\beta]\alpha}\right) \\ &\quad + \frac{2}{(d-1)(d-2)} R g_{\mu[\alpha}g_{\beta]\nu} \end{aligned} \tag{6}$$

Introduction

- Examples: *S. Deser (1993)*
  1. $d = 2$
  $$g^{\mu\nu}\langle T_{\mu\nu}\rangle = aE_2 \tag{7}$$
  2. $d = 4$
  $$g^{\mu\nu}\langle T_{\mu\nu}\rangle = aE_4 + c\sqrt{g}W_{\mu\nu\alpha\beta}W^{\mu\nu\alpha\beta} \tag{8}$$
  3. $d = 6$
  $$\begin{aligned} g^{\mu\nu}\langle T_{\mu\nu}\rangle &= aE_6 + c_1\sqrt{g}(W^{\mu\nu}_{\rho\sigma}W^{\rho\sigma}_{\alpha\beta}W^{\mu\nu}_{\alpha\beta} \\ &+ c_2 W^{\mu\nu}_{\rho\sigma}W^{\rho\alpha}_{\mu\beta}W^{\sigma\beta}_{\nu\alpha} \\ &+ c_3 W_{\mu\nu\alpha\beta}\left\{\Box\delta^{\mu}_{\gamma} - 4R^{\mu}_{\gamma} + \frac{6}{5}R\delta^{\mu}_{\gamma}\right\}W^{\gamma\nu\alpha\beta}) \end{aligned} \tag{9}$$
- $d =$odd: $g^{\mu\nu}\langle T_{\mu\nu}\rangle = 0$

Introduction

- There are many applications of conformal anomalies: Hawking effect, super symmetric theories, Maxwell/Yang - Mills ($d = 4$),string theory,...
- Anomalies can be *tracked* from heat kernel expansions.
- Our work was about GJMS P_{2k} operators $\rightarrow$ higher derivatives scalar fields *Graham, Jenne, Mason & Sparling (1992)*.
- Motivation : They appears in some physical problems $\rightarrow$ for spin 1 and higher spin gauge theories, residual field are P_{2k}.
- P_2 = Yamabe operator (conformal laplacian)

$$P_2 = Y = \Delta - \frac{d-2}{4(d-1)}R \tag{10}$$

Introduction

- How to work? Try certain geometries to help computation.
- Track heat kernel coefficients to find trace anomaly numbers.
- Compare with P_2 anomaly coefficients in different dimensions.

$$\begin{array}{c|r} d=2 & a \\ d=4 & a, c \\ d=6 & a, c_1, c_2, c_3 \end{array} \tag{11}$$

Contents

Heat - kernel

- Heat - kernel technique of a operator D is defined by

$$\mathcal{K}(t;x,y;D) = \langle x|\exp(-tD)|y\rangle \qquad (12)$$

and it satisfies the equation

$$\begin{aligned}(\partial_t + D_x)\mathcal{K}(t;x,y;D) &= 0 \\ \mathcal{K}(0;x,y;D) &= \delta(x,y)\end{aligned} \qquad (13)$$

- It has an asymptotic expansion :

$$e^{-Dt} = \sum_{i=0}^{\infty} c_i t^i \qquad (14)$$

where c_i are the heat kernel coefficients which depends of geometrical quantities.

Heat - kernel

- Conformal scalar case:

$$c_2 = \frac{1}{6} R \tag{15}$$

$$c_4 = \frac{1}{360}\left(2R_{ijkl}R^{ijkl} - 2R_{ij}R^{ij} + 5R^2 - 12\nabla^2 R\right) \tag{16}$$

- These coefficients can be written in terms of Weyl tensor, Ricci scalar, Schouten tensor, Schouten scalar, or any quantity that can help to compute the anomaly.
- For tracking trace anomaly we will use another basis: Q - curvature, Weyl invariants and Fefferman - Graham invariants

$$Q = \Delta R + \frac{d^3 - 4d^2 + 16d - 16}{4(d-1)(d-2)^2} R^2 - \frac{8(d-1)}{(d-2)^2} R_{ij}R^{ij} \tag{17}$$

Contents

GJMS operators in 4D

- 4D: Type - A, Type - B anomalies.

$$\langle T \rangle \sim aE_4 + bW^2 \tag{18}$$

- Can be tracked in the “usual” and holographic way.
- Holography: there’s relation between quantities in $5 \to 4$ dimensions.

$$\begin{pmatrix} \hat{W}^2 \\ \hat{\Phi}_5 \end{pmatrix} \to \begin{pmatrix} W^2 \\ \Phi_4 \end{pmatrix} \tag{19}$$

Contents

GJMS operators in 6D

- Certainly, there is no heat - kernel (known) for GJMS.
- In some geometries GJMS can be expressed as a product of "shifted Yamabes"

$$P_{2k} = \prod_{i=0}^{k-1} \left[-\nabla^2 + \tilde{c}_i R\right] \text{ in Einstein manifolds} \qquad (20)$$

$$\tilde{c}_i = \frac{(d+2i)(d-2-2i)}{4d(d-1)}$$

$$P_{2k} = \prod_{i=0}^{k-1} \left[Y - \frac{i(i+1)}{d(d-1)} R\right] \qquad (21)$$

A. Juhl (2009)

GJMS operators in 6D

- 1-term expansion

$$\mathcal{K}_i = \exp\left(Y - \frac{i(i+1)}{30}R\right) \tag{22}$$

$$= \left(\sum_i b_i t^i\right)\left\{1 - \frac{i(i+1)}{30}Rt + \frac{i^2(i+1)^2}{2\cdot 30^2}R^2t^2 + \frac{i^3(i+1)^3}{3!\cdot 30^3}R^3t^3 + \ldots\ldots\right\}$$

- Anomalies will be related to logarithmic term $\rightarrow$ seek t^3 terms.

GJMS operators in 6D

- t^3 terms are

$$b_6 + b_4 \frac{i(i+1)}{30} R + \frac{i^2(i+1)^2}{2 \cdot 30^2} R^2 b_2 + \frac{i^3(i+1)^3}{3!30^3} R^3 \quad (23)$$

- In this geometry b_2 and b_4 are written easily in terms of Weyl and Ricci scalar

$$(4\pi)^3 \cdot 180 b_4 = W_{abcd} W^{abcd} \sim W^2 \quad (24)$$

$$(4\pi)^3 b_2 = -\frac{1}{30} R \quad (25)$$

GJMS operators in 6D

- All possible factors have to be written in terms of $(Q, W^3, \tilde{W}^3, W\Box W)$

$$
\begin{aligned}
I_1 &= W^{\mu\nu}_{\rho\sigma} W^{\rho\alpha}_{\mu\beta} W^{\sigma\beta}_{\nu\alpha} \\
I_2 &= W^{\mu\nu}_{\rho\sigma} W^{\rho\sigma}_{\alpha\beta} W^{\mu\nu}_{\alpha\beta} \\
I_3 &= W_{\mu\nu\alpha\beta} \Box W^{\mu\nu\alpha\beta}
\end{aligned}
$$

- *F. Bastianelli, S. Frolov, A. A. Tseytlin (2000)*

$$
\begin{aligned}
b_4 &\rightarrow (0, 4, -1, -1) \\
b_6 &\rightarrow \left(-10, -8, 10, \tfrac{13}{3}\right) \\
b_2 &\rightarrow (1, 0, 0, 0)
\end{aligned}
\tag{26}
$$

GJMS operators in 6D

- All together

$$\begin{aligned}(4\pi)^3\langle T\rangle &= 7\left(\frac{k^7}{7}-k^5+\frac{4}{3}k^3\right)Q_6 \\ &+ \frac{14}{3}(k^3-k)\left(-4I_1+I_2+I_3\right) \\ &+ k\left(\frac{8}{3}I_1-\frac{10}{3}I_2-\frac{13}{9}I_3\right)\end{aligned} \tag{27}$$

- For $k=1$, the result check trace anomalies for conformal laplacian.

Contents

Holographic computation

- Here we recall the *holographic formula* from the AdS/CFT prescription:

$$-\log \frac{\det_{+}\left(-\hat{\nabla}^2 + m_k^2\right)}{\det_{-}\left(-\hat{\nabla}^2 + m_k^2\right)}\Bigg|_{X^{d+1}} = \log\det\left(P_{2k}\right)\big|_{M^d} \tag{28}$$

  where $M^d = \partial X^{d+1}$.
- $-\hat{\nabla}^2 + m_k^2$ is a massive scalar field in the bulk but setting $m_k^2 = -\frac{d^2}{4} + k^2$,massive scalar field is equal to dual of GJMS.

Holographic computation

- Idea: Einstein bulk with Einstein boundary:

$$\hat{g} = \frac{dr^2 + (1 - \lambda r^2)^2 g}{r^2} \tag{29}$$

with $\lambda = \frac{R}{4d(d-1)} = \frac{R}{120}$.

- Determinant computation

$$\sim \int_\epsilon \frac{dr}{r^7} \left(1 - \lambda r^2\right)^6 \int_0^\infty \frac{dt}{t} t^{-7/2} e^{-k^2 t} e^{9t} e^{\hat{\nabla}^2 t}$$

$$\sim \quad e^{-k^2 t} \left(1 + 9t + \frac{81}{2} t^2 + \frac{243}{2} t^3 + \ldots\right) \tag{30}$$
$$\left(1 + \hat{a}_2 t + \hat{a}_4 t^2 + \hat{a}_6 t^3 + \ldots\right)$$

Holographic computation

- Heat kernel coefficients of $\hat{\nabla}^2$ depends of $\hat{R}, \hat{R}_{\mu\nu}, \hat{R}_{\mu\nu\alpha\beta}$ so they can be written in terms of Weyl tensor (and contractions).
- Proper time integration $\rightarrow$ Weyl dependance of terms (including Weyl invariants) but in 7D.
- To find anomaly coefficients of boundary field we have to find logarithmic term (in r) with the corresponding factors

$$\begin{gathered}\frac{(1-\lambda r^2)^6}{r^7} \rightarrow \frac{Q_6}{r} \\ \frac{(1-\lambda r^2)^6}{r^7} \hat{W}^2 \rightarrow \frac{RW^2}{r} \\ \frac{(1-\lambda r^2)^6}{r^7} \hat{I}_{1,2} \rightarrow \frac{I_{1,2}}{r}\end{gathered} \tag{31}$$

Holographic computation

- Computing all these factors with the respective anomaly

$$
\begin{aligned}
(4\pi)^3\langle T\rangle &= 7\left(\frac{k^7}{7}-k^5+\frac{4}{3}k^3\right)Q_6 \\
&+ \frac{14}{3}(k^3-k)\left(-4I_1+I_2+I_3\right) \\
&+ k\left(\frac{8}{3}I_1-\frac{10}{3}I_2-\frac{13}{9}I_3\right)
\end{aligned} \tag{32}
$$

so holographic computation is ok! .

Holographic computation

- Comments:
  1. Conformal invariants

$$\begin{pmatrix} \hat{I}_1 \\ \hat{I}_2 \\ \hat{I}_3 \end{pmatrix} \rightarrow \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{33}$$

.

  2. Fefferman - Graham invariant

$$\hat{\Phi}_7 \rightarrow \Phi_6 \tag{34}$$

where

$$\Phi_d = |\nabla W|^2 + 16PWW$$

so these invariants can be used for holographic computations.

Contents

Conclusions & outlook

- This work allows us to compute trace anomalies of GJMS operators in 6D.
- Holographic computation is ok!
- (Q,I's, Φ's) are a convenient basis for holograhic computation.
- Maybe is a way of finding conformal anomalies of other fields with similar properties (CHS).
- 8D?

Acknowledgments to CONICYT for financial support..

and you for being here!

Many thanks!