Schwarzschild metric and Friedmann equations from Newtonian Gravitational collapse

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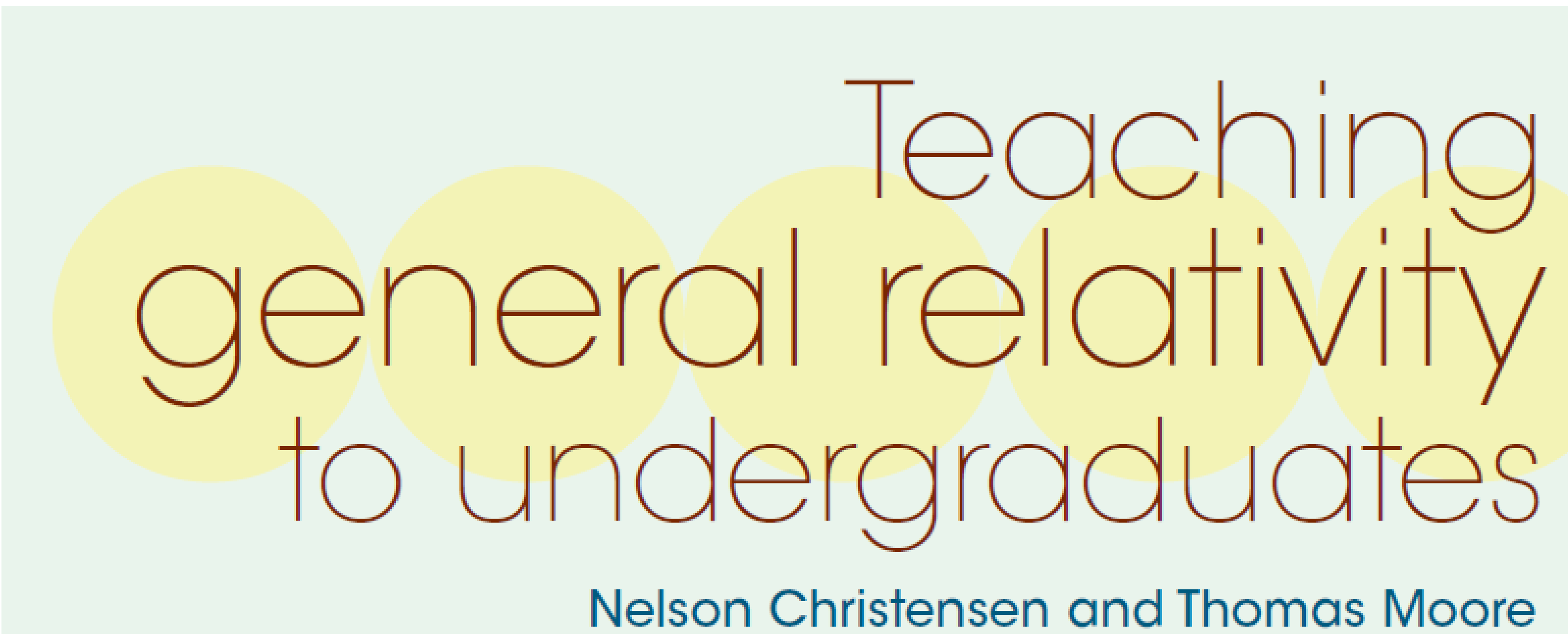
Teaching GR by elementary tools is important, see for example the article



Still, it is important not to give up

and give a derivation of the basic results, some books , propose teaching GR by giving students the Schwarzschild metric and telling the: “ok, start from there”, but it is a bad for a scientist to believe some result and continue from there.

Here we will deriving the basic results, including the Schwarzschild metric and Friedmann equations by elementary means is possible, from a study of Newtonian gravitational collapse, i.e., a system that consists of a collapsing “ball” of dust matched to an unknown static outside, to be determined precisely by the matching!.

NEVER USE EINSTEIN’S EQUATIONS!

The Principle of Equivalence

A fundamental property of gravitational fields according to *Einstein*, creates an analogy between the motion of a particle in such a field and the motion of a free particle in an accelerating or non-inertial reference frame without the presence of the field. In particular, given a set of particles, the equations which govern their motion in a non-inertial system are the same as those which govern the motion of particles in an inertial system, in the presence of a gravitational field. This is a rather loose description of the *principle of equivalence*.

First we shall look at this principle in a local fashion and "non-relativistic" velocities, here we will assume the equivalence (which was verified in numerous tests, this is actually also known as the "weak principle of equivalence") of the gravitational mass and the inertial mass. Given a system of N particles moving under the influence of some forces $\mathbf{F}(\mathbf{x_i}-\mathbf{x_j})$ and an external uniform gravitational field $\mathbf{g}$, for an observer A the motion is governed by:

$$m_i\frac{d^2\mathbf{x}_i}{dt^2}=m_i\mathbf{g}+\sum_i\mathbf{F}(\mathbf{x_i}-\mathbf{x_j}) \tag{1}$$

performing a transformation to a non-inertial coordinate system A':

$$\mathrm{x} = \mathrm{x}' + \frac{1}{2}\mathrm{g}t'^2, \qquad t = t' \tag{2}$$

we get:

$$m_i \frac{d^2\mathrm{x}'_i}{dt'^2} = \sum_j \mathbf{F}(\mathrm{x}'_\mathrm{i} - \mathrm{x}'_\mathrm{j}) \tag{3}$$

we can see that both observers A, A' will see exactly the same laws of motion for the group of particles. The above cancellation in eq.(3), of the gravitational field by moving to an accelerating system, is the core of the principle of equivalence. Actually, observer A' is regarded to be in *free fall* and for him the entire physical phenomena is as if he was in an inertial reference system. Meaning, an observer in free fall cannot, by any set of independent measurements deduce that he is in a gravitational field

But gravitational fields in general are not homogeneous, so we say that

Given a gravitational field, it is possible to choose in every point in space-time a locally inertial coordinate system, which in a small enough region the laws of physics would take the form of those in the absence of gravitation.

An inertial coordinate system is actually analogous to a cartesian or any other flat coordinate system. This formulation is consistent with the fact that the gravitational force is not considered a force at all, but actually a curvature of space-time which is smooth enough so we can regard a small enough region of it as flat, and thus without gravitation.

Metrics and Gravitation

We first discuss the specific case of a flat space-time. Given two infinitesimally close events or space-time points $[y^{\mu}, y^{\mu}+dy^{\mu}]$, this coordinate interval defines an invariant structure, the *interval:*

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{4}$$

this is the most basic form given in cartesian coordinates. We can further simplify this form by using the *Minkowski metric tensor*, which is defined by the matrix:

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

Now we can write the interval as:

$$ds^2 = \eta_{\mu\nu} dy^{\mu} dy^{\nu} \tag{6}$$

In general we can look at an interval using any other set of coordinates - x^{μ}. Assuming we have some transformation rule (which should be also reversible) $y^{\alpha} \rightarrow x^{\mu}$:

$$ds^2 = \eta_{\alpha\beta} dy^\alpha dy^\beta = \eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} dx^\mu \frac{\partial y^\beta}{\partial x^\nu} dx^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu \tag{7}$$

where we have defined the *metric tensor:*

$$g_{\mu\nu}(x) \equiv \eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \tag{8}$$

We can see that in general eq.(8) depends on the new coordinates x^μ, this is a very important property (as we shall see), which is related to the locality of a flat embedded in a curved space-time

The metric components obviously depend on the chosen local coordinate system. Under a change of coordinates $x^\mu \to x'^\mu$ the metric components transform as:

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \tag{13}$$

2.3 Relation to the principle of equivalence

As we saw in section (1), the equivalence principle states that the local properties of a curved space-time at some point, are equivalent to those of a flat space-time. Meaning, at each point X in space-time we can find some new coordinate system such that:

$$g'_{\mu\nu}(X') = \eta_{\mu\nu} \tag{14}$$

where X' is exactly the point X, but in the new coordinate system. Obviously any such locally flat coordinate system can always be transformed to a cartesian one.

2.4 Space and time properties

The interval imparts information about the causal structure between two events in space-time, according to its numerical value:

- When $ds^2 > 0$, the interval is *timelike*, meaning one can find a coordinate system in which the two events are at the same place.
- When $ds^2 < 0$, the interval is *spacelike*, meaning one can find a coordinate system in which the two events are simultaneous.
- When $ds^2 = 0$, the interval is *lightlike*, this interval can only be traversed by light, and not by any massive object.

3.1 The geodesic equation

We shall examine a free particle moving in a curved space-time denoted by the coordinates: x^μ. According to the principle of equivalence we know that there is a local flat (or freely falling) coordinates system: $y^\alpha(x)$. In this case the equation of motion for the particle is:

$$\frac{d^2 y^\alpha}{d\tau^2} = 0 \tag{19}$$

or

$$\frac{d}{d\tau}\left(\frac{\partial y^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{dy^\alpha}{dx^\mu}\frac{d^2 x^\mu}{d\tau^2} = 0 \tag{20}$$

Multiplying this $\partial x^\sigma / \partial y^\alpha$, we get the geodesic equation:

$$\frac{d^2 x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{21}$$

Where we defined the *affine connection* as:

$$\Gamma^\sigma_{\mu\nu} \equiv \frac{\partial x^\sigma}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu} \tag{22}$$

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\mu\rho}}{\partial x^{\nu}} + \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} + \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right) \tag{23}$$

3.4 The weak field approximation

Consider a slowly moving particle in a weak stationary gravitational field[2] The condition of moving slowly means that:

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \tag{28}$$

(21) can be written as

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{00} \left(\frac{dt}{d\tau} \right)^2 + 2\Gamma^{\mu}_{0i} \frac{dt}{d\tau} \frac{dx^i}{d\tau} + \Gamma^{\mu}_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0 \tag{29}$$

according to (28) the last two terms may be neglected, and we are left with:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{00} \left(\frac{dt}{d\tau} \right)^2 = 0 \tag{30}$$

since the field is stationary (and so does the metric), from (23) the affine connection takes the form:

$$\Gamma^{\mu}_{00} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\nu 0}}{\partial t} + \frac{\partial g_{0\nu}}{\partial t} - \frac{\partial g_{00}}{\partial x^{\nu}} \right) = -\frac{1}{2} g^{\mu\nu} \frac{\partial g_{00}}{\partial x^{\nu}} \tag{31}$$

The 00 component of the metric equals 1-2GM/r

because of the weakness of the field, we can decompose the metric as follows:

$$g_{\mu\nu} = \eta_{\mu\nu} + w_{\mu\nu}, \qquad |w_{\mu\nu}| \ll 1 \tag{32}$$

inserting into (31):

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial w_{00}}{\partial x^{\nu}} \tag{33}$$

dividing (30) by $(\frac{d\tau}{dt})^2$takes the geodesic equation in to the familiar "Newtonian form":

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2}\frac{\partial w_{00}}{\partial x^i} \equiv -\frac{\partial \phi}{\partial x^i} \tag{34}$$

where the perturbation to the metric is identified with the gravitational potential:

$$w_{00} = 2\phi \implies g_{00} = 1 + 2\phi \tag{35}$$

Isotropic & Homogeneous Cosmology

1. when averaged over sufficiently large scales, the observable properties of the Universe are isotropic, i.e. independent of direction;

   it remains to be clarified what *sufficiently large* scales are; nearby galaxies are very anisotropically distributed, distant galaxies approach isotropy, the microwave background is almost perfectly isotropic

2. our position in the Universe is by no means preferred to any other (cosmological principle);

   reflects Copernican revolution of the world model, when it was realised that the Earth is not at the centre of the Universe;

Co-moving observers in cosmology

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

spatial coordinates attached to fundamental observers are called comoving coordinates; in such coordinates, $dx^i = 0$ for fundamental observers; requiring that their eigentime equal the coordinate time dt, we have

$$ds^2 = g_{00} dt^2 = c^2 dt^2 \;\Rightarrow\; g_{00} = c^2 \tag{1.2}$$

- isotropy requires that clocks can be synchronised such that $g_{0i} = 0$; if that was impossible, the components of g_{0i} singled out a preferred direction in space, violating isotropy; thus

$$g_{0i} = 0 \tag{1.3}$$

A positive spatial curvature cosmology

by considering first an embedding four dimensional

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2$$

define the 3-sphere as the sets of points (x, y, z, w) which satisfy

$$x^2 + y^2 + z^2 + w^2 = 1/\kappa \; with \; \kappa > 0$$

defining $r^2 = x^2 + y^2 + z^2$,

$z = rcos\theta$, $x = rsin\theta cos\phi$ and $y = rsin\theta sin\phi$, we obtain the metric of the 3-sphere,

$$ds_3^2 = [\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2]$$

$$d\Omega^2 = (d\theta^2 + sin^2\theta d\phi^2)$$

we add time and

a scale factor that multiplies the 3-sphere, line element in FRW space becomes,

$$ds^2 = -dt^2 + R(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right)$$

We assume again that that we have

a co-moving observer which satisfies $r = const.$

in the FLRW space we can use everywhere

the barred radius $\bar{r} = R(t)r$, which means $r = \frac{\bar{r}}{R(t)}$.

equation of motion for a test mass m located on the boundary of such a sphere shall be described in terms of a homogeneous positive parameter $R(t)$, where the coordinate of each particle expands according to $a(t) = constant \cdot R(t)$, where the constant depends on the particular particle therefore such equation reads

$$\ddot{a} = -\frac{G}{a^2}\left(\frac{4\pi}{3}a^3\rho\right) = -\frac{4\pi G}{3}a\rho. \tag{7}$$

which implies a similar equation for the universal expansion factor $R(t)$

$$\ddot{R} = -\frac{G}{R^2}\left(\frac{4\pi}{3}R^3\rho\right) = -\frac{4\pi G}{3}R\rho. \tag{8}$$

Basically, this corresponds to Friedmann's second equation without a cosmological constant Λ and zero pressure. As the linear dimensions scale by $R(t)$, all co-moving volumes should scale by $R(t)^3$, that is a $1/R^3$ dependency for the density, which dilutes the matter as the sphere expands.

We will be mostly interested in the pressureles fluid, p=0

In this case, as one might naively guess the matter will be diluted as the universe expands and the density of matter should go as the inverse 3rd power of the expansion factor R(t).

If we still want to include pressure in a quasi Newtonian framework, we can still proceed but here we will avoid this issue, we can integrate and get (k is integration constant , opposite in sign to the “Newtonian Energy”)

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2}$$

In special cases we verify that the geometric interpretation of k is correct,

- Just assume that flat space is a solution of the equations of gravity when there is no matter, then it turns out that flat space can be writen in the form of a cosmology (the Milne universe), with $R(t) = t$ and $k=-1$ (in units where $c=1$), this fixes the geometrical interpretation in terms of a Robertson Walker metric of the conserved Newtonian energy as the spatial curvature of space!. But we can do better,

Cloud of dust, p=0

$\rho(t)R^3(t)$ is constant. We normalize the radial coordinate r so that

$$R(0) = 1 \tag{11.9.17}$$

and therefore

$$\rho(t) = \rho(0)R^{-3}(t) \tag{11.9.18}$$

The field equations (11.9.14) or (11.9.15) and (11.9.11) are now ordinary differential equations:

$$-2k - \ddot{R}(t)R(t) - 2\dot{R}^2(t) = -4\pi G\rho(0)R^{-1}(t) \tag{11.9.19}$$

$$\ddot{R}(t)R(t) = -\frac{4\pi G}{3}\rho(0)R^{-1}(t) \tag{11.9.20}$$

We can eliminate $\ddot{R}(t)$ by adding these two equations, and find

$$\dot{R}^2(t) = -k + \frac{8\pi G}{3}\rho(0)R^{-1}(t) \tag{11.9.21}$$

Equations (11.9.19) and (11.9.20) can be recovered from (11.9.21) and its time derivative, so we can forget about them and simply use (11.9.21) to calculate $R(t)$.

We shall now assume that the fluid is at rest (in standard coordinates) at $t = 0$, so

$$\dot{R}(0) = 0 \tag{11.9.22}$$

We shall now assume that the fluid is at rest $t = 0$, so

$$\dot{R}(0) = 0$$

and therefore (11.9.21) and (11.9.17) give

$$k = \frac{8\pi G}{3}\,\rho(0)$$

Thus Eq. (11.9.21) can be written

$$\dot{R}^2(t) = k[R^{-1}(t) - 1]$$

The solution is given by the parametric equations of a *cycloid*:

$$t = \left(\frac{\psi + \sin\psi}{2\sqrt{k}}\right)$$

$$R = \tfrac{1}{2}(1 + \cos\psi) \tag{11.9.25}$$

Note that $R(t)$ vanishes when $\psi = \pi$, and hence when $t = T$, where

$$T = \frac{\pi}{2\sqrt{k}} = \frac{\pi}{2}\left(\frac{3}{8\pi G\rho(0)}\right)^{1/2} \tag{11.9.26}$$

Thus a fluid sphere of initial density $\rho(0)$ and zero pressure will collapse from rest to a state of infinite proper energy density in the finite time T.

Cut this solution at a certain fixed comoving sphere (r=constant), assume

outside there is a generic spherically symmetric metric, which without loosing generality can be written as

$$d\tau^2 = \left(1 - \frac{2MG}{\bar{r}}\right) d\bar{t}^2$$

$$- \ A \ d\bar{r}^2 - \bar{r}^2 \, d\bar{\theta}^2 - \bar{r}^2 \sin^2 \bar{\theta} \, d\bar{\varphi}^2$$

But now we have to match this to the interior using the same coordinates!. A is to be found.

Notice that we assume that the external metric is time independent.

A radially falling geodesic, is fully described by conservation of energy from the action

$$S = \int d\sigma \sqrt{-\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} g_{\mu\nu}(x)}$$

The equation with respect to $\bar{t}$ is

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{\bar{t}}}\right) = 0 \quad where \quad \dot{\bar{t}} = \frac{d\bar{t}}{d\sigma}$$

$$\gamma = \frac{\partial L}{\partial \dot{\bar{t}}} = \left(1 - \frac{2GM}{\bar{r}}\right)\frac{d\bar{t}}{d\tau}$$

where γ is constant and $d\tau$ is the proper time

the FLRW metric in the space $dx^i = 0$ we get

$$d\tau = dt$$

$$d\tau^2 = \left(1 - \frac{2GM}{\bar{r}}\right) d\bar{t}^2 - A(\bar{r}) d\bar{r}^2$$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 = \left(1 - \frac{2GM}{\bar{r}}\right) \left(\frac{d\bar{t}}{dt}\right)^2 - A(\bar{r}) \left(\frac{d\bar{r}}{dt}\right)^2$$

we obtain

$$\left(1 - \frac{2GM}{\bar{r}}\right)^{-1} \gamma^2 - A(\bar{r}) \left(\frac{d\bar{r}}{dt}\right)^2 = 1$$

the consistency of the matching

of the two spaces requires $\bar{r} = R(t)r$,

This is because if we move around a circle in the boundary, we must get the

Same answer from the inside and from the outside. This implies a relation between the coordinates:

coordinate $\bar{r}$, $\bar{\theta}$, $\bar{\varphi}$ must be chosen as

$$\bar{r} = rR(t), \qquad \bar{\theta} = \theta, \qquad \bar{\varphi} = \varphi$$

furthermore, we assume that even the boundary of the dust shell free falls according to a co-moving observer, which means that the FLRW coordinate $r = constant$ and this allow then to solve for $A(\bar{r})$,

$$A(\bar{r}) = -\left(1 - \frac{\gamma^2}{\left(1 - \frac{2GM}{\bar{r}}\right)}\right) \frac{1}{r^2 \left(\frac{dR}{dt}\right)^2}$$

$$\dot{R}(t)^2 = k\left[\frac{1}{R} - 1\right]$$

and get

$$A(\bar{r}) = -\left(1 - \frac{\gamma^2}{\left(1 - \frac{2GM}{\bar{r}}\right)}\right)\frac{1}{r^2 k\left(\frac{1}{R} - 1\right)}$$

and expressing in terms of $\bar{r}$ we get

$$A(\bar{r}) = \frac{1}{r^2}\frac{\gamma^2 - 1 + \frac{2GM}{\bar{r}}}{1 - \frac{2GM}{\bar{r}}}\frac{1}{k\left(-1 + \frac{r}{\bar{r}}\right)}$$

If we take the limit $\bar{r} \to \infty$,

we see that $A(\bar{r}) \to -(\gamma^2 - 1)/kr^2$.

Asymptotic flatness would require $A(\bar{r}) \to 1$

$\gamma^2 - 1 = -kr^2$ The metric preserves its signature only if

$$\frac{2GM}{\bar{r}} = \frac{r^2 k}{R} = \frac{kr^3}{\bar{r}} \Rightarrow k = \frac{2GM}{r^3}$$

$k = \frac{2GM}{r^3}$, when combined with

$k = \frac{8\pi G}{3}\rho(0)$ tells us $M = \frac{4}{3}\pi\rho_0 r^3$

Finally, all of this gives us

$$A(\bar{r}) = \frac{1}{1 - \frac{2GM}{\bar{r}}}$$

Reproducing the Schwarzschild spacetime.

FINDING THE GEOMETRIC INTERPRETATION OF THE NEWTONIAN COSMOLOGY

to demonstrate that $k = \kappa$.

$$r = \frac{\bar{r}}{R(t)}. \quad dr = \frac{d\bar{r}}{R(t)} - \frac{\dot{R}\bar{r}}{R^2}dt$$

$$ds^2 = \left(-1 + \frac{\dot{R}^2\bar{r}}{(1-\kappa r^2)R^2}\right)dt^2 + \frac{d\bar{r}^2}{1-\kappa r^2} - \frac{2\dot{R}\bar{r}}{(1-\kappa r^2)R}dtd\bar{r}$$

$t = t(\bar{t}, \bar{r})$, so infinitesimal change in time

$$dt = \frac{\partial t}{\partial \bar{r}} d\bar{r} + \frac{\partial t}{\partial \bar{t}} d\bar{t}$$

$$dt^2 = \left(\frac{\partial t}{\partial \bar{r}}\right)^2 d\bar{r}^2 + \left(\frac{\partial t}{\partial \bar{t}}\right)^2 d\bar{t}^2 + 2\left(\frac{\partial t}{\partial \bar{r}}\right)\left(\frac{\partial t}{\partial \bar{t}}\right) d\bar{t}\,d\bar{r}$$

$$ds^2 = \left(-1 + \frac{\dot{R}^2 \bar{r}}{(1-\kappa r^2)R^2}\right)\left(\left(\frac{\partial t}{\partial \bar{r}}\right)^2 d\bar{r}^2 + \left(\frac{\partial t}{\partial \bar{t}}\right)^2 d\bar{t}^2 + 2\left(\frac{\partial t}{\partial \bar{r}}\right)\left(\frac{\partial t}{\partial \bar{t}}\right) d\bar{t}\,d\bar{r}\right)$$
$$+ \frac{d\bar{r}^2}{1-\kappa r^2} - \frac{2\dot{R}\bar{r}}{(1-\kappa r^2)\,R} dr \left(\frac{\partial t}{\partial \bar{r}} d\bar{r} + \frac{\partial t}{\partial \bar{t}} d\bar{t}\right)$$

have to eliminate the cross terms, so

$$-\frac{2\dot{R}\bar{r}}{(1-\kappa r^2)R}dr\frac{\partial\bar{t}}{\partial\bar{t}}d\bar{t}+\left(-1+\frac{\dot{R}^2\bar{r}}{(1-\kappa r^2)R^2}\right)2\left(\frac{\partial t}{\partial\bar{r}}\right)\left(\frac{\partial t}{\partial\bar{t}}\right)d\bar{t}d\bar{r}=0$$

$$\begin{aligned}
\left(\frac{\partial t}{\partial \bar{r}}\right) &= \frac{\bar{r}\dot{R}}{(1-\kappa r^2)R\left(-1+\frac{\dot{R}^2\bar{r}}{(1-\kappa r^2)R^2}\right)} \\
&= \frac{\bar{r}\dot{R}}{-(1-\kappa r^2)R+\frac{\dot{R}^2\bar{r}^2}{R}} \\
&= \frac{\bar{r}\dot{R}R}{-(1-\kappa r^2)R^2+\dot{R}^2\bar{r}^2}
\end{aligned}$$

For $\bar{r} = R(t)r$ it yields

$$g_{\bar{r}\bar{r}} = \frac{1}{1 - r^2(\kappa + \dot{R}^2)}$$

we derived also what this metric component should be

$$g_{\bar{r}\bar{r}} = \frac{1}{1 - \frac{2GM}{\bar{r}}}$$

from Newtonian cosmology , we know that

$$k + \dot{R}^2 = \frac{8\pi}{3R} G\rho_0 = \frac{8\pi r^3 \rho_0}{3r^2(rR)}$$

Therefore,

$$(k + \dot{R}^2) r^2 = \frac{8\pi r^3 \rho_0}{3Rr} = \frac{2GM}{\bar{r}} = r^2(\kappa + \dot{R}^2)$$

where we have used first that $M = \frac{4}{3}\pi\rho_0 r^3$ and $\bar{r} = rR$

and afterwards demanded that the

two expressions for $g_{\bar{r}\bar{r}}$ above must agree.

This gives us the relation $k = \kappa$.

OK, so we got the Schwarzschild solution without ever invoking the

Einstein's equations!. The steps were:

1. Reinterpreting the Newtonian results as

geodesic motion, this gave us the 00 component of the metric generated by a source.

2. From Newtonian Dynamics, develop cosmological equations, they can be related to Robertson Walker space times, however at first the relation between the Newtonian energy and spatial curvature is not clear.

3. The Schwarzschild solution is obtained as the exterior solution of a collapsing dust shell (obtained in 2) DEMANDING GEODESICS OF INNER AND OUT SIDE SPACES AGREE AT THE CO-MOVING, FREE FALLING BOUNDARY

4. We can find the geometric meaning

of the Newtonian energy in the Newtonian cosmology, by expressing internal space metric and external space metric in the same coordinates and identifying the rr components, this then establishes that the Newtonian energy in the Newtonian cosmology relates to the spacial curvature of the FLRW space, thus establishing the Friedmann equations, without using GR.

THANK YOU FOR YOUR ATTENTION!

Reference: paper with my students in “Gravity 2 “ course last semester,

Schwarzschild and Friedmann Lema\^itre Robertson Walker metrics from Newtonian Gravitational collapse

Eduardo I. Guendelman, Arka Prabha Banik, Gilad Granit, Tomer Ygael, Christian Rohrhofer. Jan 27, 2015. 15 pp.

e-Print: **arXiv:1501.06762 [gr-qc]**

FURTHER MATHEMATICAL NOTES, AFTER DEFINING THE BAR RADIOUS, ALSO IN THE FRIEDMANN, LEMAITRE, ROBERSON WALKER COSMOLOGY WE CAN GENERICALLY A BAR TIME

define a standard time coordinate such that $d\tau^2$ does not contain a

cross-term $d\bar{r}\, d\bar{t}$,

$$\bar{t} = \left(\frac{1 - ka^2}{k}\right)^{1/2} \int_{S(r,t)}^{1} \frac{dR}{(1 - ka^2/R)} \left(\frac{R}{1 - R}\right)^{1/2}$$

$$S(r, t) = 1 - \left(\frac{1 - kr^2}{1 - ka^2}\right)^{1/2} (1 - R(t))$$

We find now that the inside metric takes the form

$$d\tau^2 = B(\bar{r}, \bar{t})\, d\bar{t}^2 - A(\bar{r}, \bar{t})\, d\bar{r}^2 - \bar{r}^2(d\bar{\theta}^2 + \sin^2 \bar{\theta}\, d\bar{\varphi}^2)$$

$$B = \frac{R}{S}\left(\frac{1 - kr^2}{1 - ka^2}\right)^{1/2} \frac{(1 - ka^2/S)^2}{(1 - kr^2/R)}$$

$$A = \left(1 - \frac{kr^2}{R}\right)^{-1}$$

a is an integration constant and at the

matching surface the coordinate r is a constant (a comoving observer), in order that the inside solution matches smoothly the 00 component of the metric, we must have that a=r. This gives us

$$B(\bar{a},\bar{t}) = \left(1 - \frac{ka^2}{R(t)}\right)$$

$$A(\bar{a},\bar{t}) = \left(1 - \frac{ka^2}{R(t)}\right)^{-1}$$

exterior solutions fit continuously at $\bar{r} = aR(t)$ if

$$d\tau^2 = \left(1 - \frac{2MG}{\bar{r}}\right) d\bar{t}^2 - \left(1 - \frac{2MG}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2\, d\bar{\theta}^2 - \bar{r}^2 \sin^2 \bar{\theta}\, d\bar{\varphi}^2$$

$$k = \frac{2MG}{a^3}$$

this just says that

$$M = \frac{4\pi}{3}\rho(0)a^3$$

Problem Set1: 1, show that for the F-R-W space for k>0, to be matched to the Schwarzschild space , then new F-R-W

coordinate $\bar{r}$, $\bar{\theta}$, $\bar{\varphi}$ must be chosen as

$$\bar{r} = rR(t), \qquad \bar{\theta} = \theta, \qquad \bar{\varphi} = \varphi$$

define a standard time coordinate such that $d\tau^2$ does not contain a

cross-term $d\bar{r}\,d\bar{t}$,

$$\bar{t} = \left(\frac{1 - ka^2}{k}\right)^{1/2} \int_{S(r,t)}^{1} \frac{dR}{(1 - ka^2/R)} \left(\frac{R}{1 - R}\right)^{1/2}$$

$$S(r,t) = 1 - \left(\frac{1 - kr^2}{1 - ka^2}\right)^{1/2} (1 - R(t))$$

We find now that the inside metric takes the form

$$d\tau^2 = B(\bar{r},\bar{t})\, d\bar{t}^2 - A(\bar{r},\bar{t})\, d\bar{r}^2 - \bar{r}^2(d\bar{\theta}^2 + \sin^2\bar{\theta}\, d\bar{\varphi}^2)$$

$$B = \frac{R}{S}\left(\frac{1 - kr^2}{1 - ka^2}\right)^{1/2} \frac{(1 - ka^2/S)^2}{(1 - kr^2/R)}$$

$$A = \left(1 - \frac{kr^2}{R}\right)^{-1}$$

a is an integration constant and at the

matching surface the coordinate r is a constant (a comoving observer), in order that the inside solution matches smoothly the 00 component of the metric, we must have that a=r. This gives us

$$B(\bar{a}, \bar{t}) = \left(1 - \frac{ka^2}{R(t)}\right)$$

$$A(\bar{a}, \bar{t}) = \left(1 - \frac{ka^2}{R(t)}\right)^{-1}$$

exterior solutions fit continuously at $\bar{r} = aR(t)$ if

$$d\tau^2 = \left(1 - \frac{2MG}{\bar{r}}\right) d\bar{t}^2 - \left(1 - \frac{2MG}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2\, d\bar{\theta}^2 - \bar{r}^2 \sin^2 \bar{\theta}\, d\bar{\varphi}^2$$

$$k = \frac{2MG}{a^3}$$

this just says that

$$M = \frac{4\pi}{3} \rho(0) a^3$$