GENERATING INFINITE-DIMENSIONAL ALGEBRAS FROM LOOP ALGEBRAS BY EXPANDING MAURER-CARTAN FORMS

R. Caroca[1,2], N. Merino[1], P. Salgado[1] and O. Valdivia[1]

[1]Departamento de Física, Universidad de Concepción Casilla 160-C, Concepción, Chile

[2] Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Alonso de Rivera 2850, Concepción, Chile

rcaroca@ucsc.cl

GENERATING INFINITE-DIMENSIONAL ALGEBRAS FROM LOOP ALGEBRAS BY EXPANDING MAURER-CARTAN FORMS

- Invariance principles and the construction of invariant actions under certain Lie algebras are of great importance in physics.
- The construction of the new Lie algebras from other Lie algebras is an interesting problem for Physics and Mathematics.

GENERATING INFINITE-DIMENSIONAL ALGEBRAS FROM LOOP ALGEBRAS BY EXPANDING MAURER-CARTAN FORMS

It is shown that the expansion procedure of the Maurer-Cartan 1-forms, developed in de Azcarraga *et al. (dAIVP)* can be generalized so that they permit to study the expansion of algebras of loops, both when the compact finite-dimensional algebra and the algebra of loops have a decomposition into two vector subspaces.

GENERATING INFINITE-DIMENSIONAL ALGEBRAS FROM LOOP ALGEBRAS BY EXPANDING MAURER-CARTAN FORMS

INTRODUCTION

Let $\mathcal{G}(S^1) = Map(S^1; G)$ be, the group of smooth mappings (loops) $z \to g(z)$ of the circle $S^1 = \{z \in C/|z| = 1\}$ into a simple, compact, and connected finite-dimensional Lie group G. The group structure is defined by the pointwise multiplication of functions $(g'g)(z) = g'(z)g(z)$. $Map(S1; G)$ is an infinite-dimensional group, the loop group, the elements of which can be represented by

$$g(z) = e^{\alpha^a(z)T_a} \quad , \quad a = 1,\ldots,r = \dim G \tag{1}$$

where $T_a = -T_a^\dagger$ are the generators of the finite-dimensional Lie algebra $\mathcal{G}$, $[T_a, T_b] = f_{ab}^c T_c$. For elements near the identity,

$$g(z) \simeq 1 + \alpha^a(z)T_a. \tag{2}$$

Making a Laurent expansion of $\alpha^a(z)$ on the circle

$$\alpha^a(z) = \sum_{n=-\infty}^{\infty} \alpha_{-n}^a z^n \tag{3}$$

expression (2) reads

$$g(z) \simeq 1 + \sum_{n=-\infty}^{\infty} \alpha_{-n}^{a} T_a z^n = 1 + \sum_{n=-\infty}^{\infty} \alpha_{-n}^{a} T_a^n \quad , \quad T_a^n = T_a z^n \qquad (4)$$

where T_a^n are the generators of the algebra $G \equiv \mathcal{G}(S1)$. We may now write the commutation relations of the Lie algebra in terms of the generators T_a^n. The commutators of the infinite-dimensional loop algebra $\hat{\mathcal{G}}$ are

$$[T_a^m, T_b^n] = f_{ab}^{c} T_c^{m+n} \qquad (5)$$

where $T_a^{m\dagger} = -T_a^{-m}$ since, z being of unit modulus, $z^* = z^{-1}$. On the other hand, if $\{\omega^a(g)\}$, $a = 1,\ldots,r = dimG$, is the basis determined by the (dual, left-invariant) Maurer–Cartan one-forms on $\mathcal{G}$; then, the Maurer–Cartan equations that characterize $\mathcal{G}$, in a way dual to its Lie bracket description, are given by

$$d\omega^c = -\tfrac{1}{2} C_{ab}^{c} \omega^a \wedge \omega^b \quad , \quad a,b,c = 1,\ldots,r = \dim G. \qquad (6)$$

In direct analogy we can say that if $\{\omega^{a,n}(g)\}$, $a = 1,\ldots,r = dimG$, $n \in Z$ is the basis determined by the (dual, left-invariant) Maurer–Cartan one-forms on loop group; then, the corresponding Maurer–Cartan equations that characterize the algebra $\hat{\mathcal{G}}$, are given by

$$d\omega^{c,l} = -\tfrac{1}{2} f^{(c,l)}_{(a,m)(b,n)} \omega^{a,n} \wedge \omega^{b,m}, \quad a,b,c = 1,\ldots,r \; ; \; l,m,n \in \mathbb{Z}. \tag{7}$$

The purpose of this paper is to generalize the expansion procedures developedby J. Azcárraga et al., so that it permits to study the expansion of the algebrasof loops when both the compact finite-dimensional algebra $\mathcal{G}$ and the loop algebra (which is an infinite-dimensional algebra $\hat{\mathcal{G}}$) have a decomposition into two subspaces $V_0 \oplus V_1$.

RESCALING OF THE GROUP PARAMETERS AND THE EXPANSION PROCEDURE

Let LG be the loop group of local coordinates $g^a(z)$,$a = 1,\ldots,r = dimG$. Let $\hat{\mathcal{G}}$ be its algebra of basis $\{T^n_a\}$, which may be realized by left-invariant generators $T^n_a(g)$ on the group manifold. Let $\hat{\mathcal{G}}^*$ be the coalgebra, and let $\{\omega^{a,n}(g)\}$, $a = 1,\ldots,r$; $n \in Z$, be the basis (dual, i.e., $\omega^{a,n}(T_{b,m}) \equiv \delta^n_m \delta^a_b$) determined by the Maurer–Cartan one-form on LG. Then, when $[T^m_a, T^n_b] = f^c_{ab} T^{m+n}_c$, the Maurer–Cartan equations read

$$d\omega^{c,l} = -\frac{1}{2}C^{(c,l)}_{(a,m)(b,n)}\omega^{a,n}\wedge\omega^{b,m}, \quad a,b,c = 1,\ldots,r \; ; \; l,m,n\epsilon\mathbb{Z}.$$ 8

Let θ be the left-invariant canonical form on LG,

$$\theta = g^{-1}dg = e^{-ig_{a,n}T^{a,n}}de^{ig_{a,m}T^{a,m}} \equiv \omega^{a,n}T_{a,n} \quad , \quad a = 1,\ldots,r = \dim G; n\epsilon\mathbb{Z}$$ 9

Since

$$e^{-A}de^{A} = dA + \frac{1}{2!}[dA,A] + \frac{1}{3!}[[dA,A],A] + \frac{1}{4!}[[[dA,A],A],A] + \ldots$$ 10

one obtains, for $A = g_{a,n}T^{a,n}$ the expansion of $\theta(g)$ as polynomials in the group coordinates $g^{a,n}$:

$$\theta = e^{-ig_{a,n}T^{a,n}}de^{ig_{a,m}T^{a,m}} \equiv idg_{a_1,n_1}T^{a_1,n_1} + \frac{i^2}{2!}[dg_{a_2,n_2}T^{a_2,n_2}, g_{a_3,n_3}T^{a_3,n_3}]$$
$$+ \frac{i^3}{3!}[[dg_{a_2,n_2}T^{a_2,n_2}, g_{a_3,n_3}T^{a_3,n_3}], g_{a_4,n_4}T^{a_4,n_4}] + \ldots .$$ 11

where the indices $a_1, a_2, \ldots = 1, 2, \ldots, dim\mathcal{G}$, and $n_i \in Z$. Factoring the coordinates and their derivatives in the Lie brackets

$$\theta = e^{-ig_{a,n}T^{a,n}} de^{ig_{a,m}T^{a,m}} \equiv idg_{a_1,n_1}T^{a_1,n_1} + \frac{i^2}{2!}dg_{a_2,n_2}g_{a_3,n_3}[T^{a_2,n_2}, T^{a_3,n_3}]$$
$$+ \frac{i^3}{3!}dg_{a_2,n_2}g_{a_3,n_3}g_{a_4,n_4}[[T^{a_2,n_2}, T^{a_3,n_3}], T^{a_4,n_4}] + \ldots \quad (12)$$

Using the commutation relation (5) we have

$$[T^{a_2,n_2}, T^{a_3,n_3}] = if_{h_1}^{a_2,a_3} T^{h_1,n_2+n_3}$$
$$[[T^{a_2,n_2}, T^{a_3,n_3}], T^{a_4,n_4}] = i^2 f_{h_1}^{a_2,a_3} f_{h_2}^{h_1,a_4} T^{h_2,n_2+n_3+n_4}$$
$$[[[T^{a_2,n_2}, T^{a_3,n_3}], T^{a_4,n_4}], T^{a_5,n_5}] = i^3 f_{h_1}^{a_2,a_3} f_{h_2}^{h_1,a_4} f_{h_3}^{h_2,a_5} T^{h_3,n_2+n_3+n_4+n_5}$$
$$\square \quad (13)$$

so that (11) takes the form

$$\theta(g) = idg_{a,n}T^{a,n} + \frac{i^3}{2!}dg_{a_2,n_2}g_{a_3,n_3}f_a^{a_2,a_3}T^{a,n_2+n_3}$$
$$+ \frac{i^5}{3!}dg_{a_2,n_2}g_{a_3,n_3}g_{a_4,n_4}f_{h_1}^{a_2,a_3}f_a^{h_1,a_4}T^{a,n_2+n_3+n_4} + \ldots \quad (14)$$

expression that can be rewritten as

$$\theta(g) = \left[\begin{array}{|c|} \hline i dg_{a,n} + \frac{i^3}{2!}\delta_n^{(n_2+n_3)} dg_{a_2,n_2} g_{a_3,n_3} f_a^{a_2,a_3} \\ \hline + \frac{i^3}{3!}\delta_n^{(n_2+n_3+n_4)} dg_{a_2,n_2} g_{a_3,n_3} g_{a_4,n_4} f_{h_1}^{a_2,a_3} f_a^{h_1,a_4} + \ldots \\ \hline \end{array} \right] T^{a,n} \tag{15}$$

Therefore, the Maurer–Cartan one-forms, $\omega_{a,n}(g)$, as a polynomial in the coordinates of the group $g_{a,n}$ is given by

$$\omega_{a,n} = \left[\begin{array}{|c|} \hline i dg_{a,n} + \frac{i^3}{2!}\delta_n^{(n_2+n_3)} dg_{a_1,n_1} g_{a_2,n_2} f_a^{a_1,a_2} \\ \hline + \sum_{\beta=2}^{\infty} \frac{i^{2\beta+1}}{(\beta+1)!}\delta_n^{(n_2+n_3+\ldots+n_{\beta+1})} dg_{a_1,n_2} g_{a_2,n_3} \ldots \\ \hline g_{a_\beta,n_{\beta+1}} g_{a_{\beta+1},n_{\beta+2}} f_{h_1}^{a_1,a_2} f_{h_2}^{h_1,a_3} \ldots f_{h_{\beta-1}}^{h_{\beta-2},a_\beta} f_a^{h_{\beta-1},a_{\beta+1}} \\ \hline \end{array} \right] \tag{16}$$

From (16) we can see that the rescaling of some coordinates $g_{a,n}$

$$g_{a,n} \rightarrow \lambda g_{a,n} \tag{17}$$

will generate an expansion of Maurer–Cartan one-forms $\omega_{a,n}(g,\lambda)$ as a sum of one-forms $\omega_{a,n}(g,\lambda)$ on LG multiplied by the corresponding powers of λ^α of λ. This means that the expansion (17) exists and can be expressed as

$$\omega_{a,n} = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega_{a,n;\alpha} \tag{18}$$

EXPANSION OF LOOP ALGEBRAS G WHEN $G = V_0 \oplus V_1$

In this section we consider the expansion of the loop algebras $\hat{\mathcal{G}}$ when the compact finite-dimensional algebra $\mathcal{G}$ has a decomposition into two subspaces $\mathcal{G} = V_0 \oplus V_1$ and we study the conditions under which the expanded algebra closes. The case when V_0 is a subalgebra is also analyzed. We consider the splitting of $\hat{\mathcal{G}}^*$ into the sum of two vector subspaces

$$\mathcal{G}^* = V_0^* \oplus V_1^* \qquad (19)$$

V_0^*, V_1^* being generated by the Maurer–Cartan forms $\omega^{a_0,n}(g)$, $\omega^{a_1,n}(g)$ of $\hat{\mathcal{G}}^*$ with indices corresponding, respectively, to the unmodified and modified parameters,

$$g^{a_0,n} \to g^{a_0,n}, \quad g^{a_1,n} \to \lambda g^{a_1,n}, \quad a_0(a_1) = 1, \ldots \dim V_0(\dim V_1), \quad n \in \mathbb{Z}. \qquad (20)$$

In general, the series of $\omega^{a_0,n}(g) \in V_0^*$, $\omega^{a_1,n}(g) \in V_1^*$ will involve all powers of λ

EXPANSION OF LOOP ALGEBRAS G WHEN G = $V_0 \oplus V_1$

In this section we consider the expansion of the loop algebras $\hat{\mathcal{G}}$ when the compact finite-dimensional algebra $\mathcal{G}$ has a decomposition into two subspaces $\mathcal{G} = V_0 \oplus V_1$ and we study the conditions under which the expanded algebra closes. The case when V_0 is a subalgebra is also analyzed. We consider the splitting of $\hat{\mathcal{G}}^*$ into the sum of two vector subspaces

$$\mathcal{G}^* = V_0^* \oplus V_1^* \qquad (19)$$

V_0^*, V_1^* being generated by the Maurer–Cartan forms $\omega^{a_0,n}(g)$, $\omega^{a_1,n}(g)$ of $\hat{\mathcal{G}}^*$ with indices corresponding, respectively, to the unmodified and modified parameters,

$$g^{a_0,n} \to g^{a_0,n}, \quad g^{a_1,n} \to \lambda g^{a_1,n}, \quad a_0(a_1) = 1,\ldots \dim V_0(\dim V_1), \quad n \in \mathbb{Z}. \qquad (20)$$

In general, the series of $\omega^{a_0,n}(g) \in V_0^*$, $\omega^{a_1,n}(g) \in V_1^*$ will involve all powers of λ

In general, the series of $\omega^{a_0,n}(g) \in V_0^*$, $\omega^{a_1,n}(g) \in V_1^*$ will involve all powers of λ

$$\omega^{a_p,n}(g,\lambda) = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{a_p,n;\alpha}(g) = \omega^{a_p,n;0}(g) + \lambda\omega^{a_p,n;1} + \lambda^2\omega^{a_p,n;2} + \ldots, p = 0,1 \qquad 21$$

where $\omega^{a_p,n}(g,1) = \omega^{a_p,n}(g)$.

The Maurer–Cartan equations (7) for $\hat{\mathcal{G}}$ can be rewritten as

$$d\omega^{c_s,l} = -\frac{1}{2} C^{c_s,l}_{a_p,n;b_q,m} \omega^{a_p,n} \omega^{b_q,m} \quad , \quad p,q,s = 0,1 \qquad 22$$

where $a_p, b_q = 1, \ldots . \dim V_0 (\dim V_1); l,n,m \in \mathbb{Z}$ and where

$$\omega^{c_s,l} = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{c_s,l;\alpha}, \quad \omega^{a_p,n} = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{a_p,n;\alpha}, \quad \omega^{b_q,m} = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{b_q,m;\alpha} \qquad 23$$

Introducing into the Maurer-Cartan equations, we obtain

$$d\omega^{c_s,l;\alpha} = -\frac{1}{2} C^{(c_s,l;\alpha)}_{(a_p,n;\beta)(b_q,m;\gamma)} \omega^{a_p,n;\beta} \omega^{b_q,m;\gamma}$$

$$C^{(c_s,l;\alpha)}_{(a_p,n;\beta)(b_q,m;\gamma)} = \delta^{\alpha}_{\beta+\gamma} C^{c_s,l}_{a_p,n;b_q,m} \qquad 24$$

where $a_p, b_q, c_s = 1, \ldots, \dim \mathcal{G}$, $l,n,m \in \mathbb{Z}$ and $\alpha, \beta = 0, 1, \ldots .$

Now we ask, under which conditions the one-forms $\omega^{a_0,l;\alpha_0}$, $\omega^{a_1,l;\alpha_1}$ generate new infinite dimensional algebras ?. The answer is given by the following analysis: consider the one-forms

$$\{\omega^{a_0,l;\alpha_0}, \omega^{a_1,l;\alpha_1}\} = \{\omega^{a_0,l;0}, \omega^{a_0,l;1}, \ldots, \omega^{a_0,l;N_0}; \omega^{a_1,l;0}, \ldots, \omega^{a_1,l;N_1}\} \quad (25)$$

with $\alpha_0 = 0,\ldots,N_0$, $\alpha_1 = 0,\ldots,N_1$, $l \in \mathbb{Z}$. The conditions under which these forms generate new algebras are found by demanding that the algebra generated by eq. (25) is closed under the exterior derivative d and that the Jacobi identities for the new algebra are satisfied.

In fact, to find the conditions under which the algebra is closed, we write:

$$\begin{aligned} d\omega^{c_0,l;N_0} = &-\frac{1}{2}c^{c_0,l}_{a_0,n\ b_0,m}\left[\omega^{a_0,n;0}\omega^{b_0,m;N_0} + \ldots + \omega^{a_0,n;N_0}\omega^{b_0,m;0}\right] \\ &-\frac{1}{2}c^{c_0,l}_{a_0,n\ b_1,m}\Big[\omega^{a_0,n;0}\underbrace{\omega^{b_1,m;N_0}}_{(i)} + \ldots + \omega^{a_0,n;N_0}\omega^{b_1,m;0}\Big] \\ &-\frac{1}{2}c^{c_0,l}_{a_1,n\ b_0,m}\Big[\omega^{a_1,n;0}\omega^{b_0,m;N_0} + \ldots + \underbrace{\omega^{a_1,n;N_0}}_{(ii)}\omega^{b_0,m;0}\Big] \\ &-\frac{1}{2}c^{c_0,l}_{a_1,n\ b_1,m}\Big[\omega^{a_1,n;0}\underbrace{\omega^{b_1,m;N_0}}_{(iii)} + \ldots + \underbrace{\omega^{a_1,n;N_0}}_{(iv)}\omega^{b_1,m;0}\Big] \end{aligned} \quad (26)$$

$$d\omega^{c_1,l;N_1} = -\frac{1}{2}c^{c_1,l}_{a_0,n\ b_0,m}\left[\omega^{a_0,n;0}\ \underset{(v)}{\omega^{b_0,m;N_1}} + \ldots + \underset{(vi)}{\omega^{a_0,n;N_1}}\ \omega^{b_0,m;0}\right]$$

$$-\frac{1}{2}c^{c_1,l}_{a_0,n\ b_1,m}\left[\omega^{a_0,n;0}\omega^{b_1,m;N_1} + \ldots + \underset{(vii)}{\omega^{a_0,n;N_1}}\ \omega^{b_1,m;0}\right]$$

$$-\frac{1}{2}c^{c_1,l}_{a_1,n\ b_0,m}\left[\omega^{a_1,n;0}\ \underset{(viii)}{\omega^{b_0,m;N_1}} + \ldots + \omega^{a_1,n;N_1}\omega^{b_0,m;0}\right]$$

$$-\frac{1}{2}c^{c_1,l}_{a_1,n\ b_1,m}[\omega^{a_1,n;0}\omega^{b_1,m;N_1} + \ldots + \omega^{a_1,n;N_1}\omega^{b_1,m;0}]. \quad 27$$

Where from we can see that the 1-forms $\omega^{b_1,m;N_0}$ and $\omega^{a_1,n;N_0}$, corresponding to the terms identified by the symbols $(i),(ii),(iii)$ and (iv) in the equation (26), belong to the base (25) if and only if

$$N_0 \leq N_1. \quad 28$$

On the other hand, the 1-forms $\omega^{b_0,m;N_1}$ and $\omega^{a_0,n;N_1}$, corresponding to the terms identified by the symbols $(v),(vi),(vii)$ and $(viii)$ in the equación (27), belong to the base (25) if and only if

$$N_1 \leq N_0. \quad 29$$

From (28) y (29) it follows trivially that the conditions under which the expanded algebra closes is

$$N_0 = N_1. \quad 30$$

The case $\hat{\mathcal{G}} = V_0 \oplus V_1$ in which V_0 is a subalgebra $L_0 \subset \hat{\mathcal{G}}$

Let $\mathcal{G} = V_0 \oplus V_1$, where now V_0 is a subalgebra $\mathcal{L}_0$ of $\mathcal{G}$. From the commutation relation

$$[T_{a,n}, T_{b,m}] = C^{c}_{ab} T_{c,n+m} = C^{c,l}_{a,n\ b,m} X_{c,l} \qquad (31)$$

$a_p, b_q = 1, \ldots, \dim V_0 (\dim V_1)$; $l, n, m \in \mathbb{Z}$. From (31) we can see that $\mathcal{L}_0 = \{T_{a,0}\}$ generates a subalgebra given by

$$[T_{a,0}, T_{b,0}] = C^{c}_{ab} X_{c,0} = C^{c,0}_{a,0\ b,0} T_{c,0}. \qquad (32)$$

From (32) we see that

$$C^{c,n}_{a,0\ b,0} = c^{c}_{ab} \delta^{n}_{0} = 0, \text{ to } n \neq 0, \quad n \in \mathbb{Z}. \qquad (33)$$

Using (33) in the expansion

$$\omega^{a,n}(g) = [\delta^{(a,n)}_{(b,m)} + \frac{1}{2!} C^{a,n}_{b,m\ c,l} g^{c,l}$$
$$+ \sum_{r=2}^{\infty} \frac{1}{(r+1)!} C^{h_1,p_1}_{b,m\ c_1,l_1} C^{h_2,p_2}_{h_1,p_1\ c_2,l_2} \cdots$$
$$\cdots C^{h_{r-1},p_{r-1}}_{h_{r-2},p_{r-2}\ c_{r-1},l_{r-1}} C^{a,n}_{h_{r-1},p_{r-2}\ c_r,l_r} g^{c_1,l_1} g^{c_2,l_2} \cdots g^{c_{r-1},l_{r-1}} g^{c_r,l_r}] dg^{b,m} \qquad (34)$$

we find that under the rescaling

$$g^{a,0} \to g^{a,0},\ g^{a,n} \to \lambda g^{a,n}\ (n \neq 0),$$
$$(a,0) = 1,\ldots,\dim V_0$$
$$(a,n) = 1,\ldots,\dim V_1.$$
$$V_1 = \{T_{a,n}\} \text{ with } n \neq 0 \qquad 35$$

the expansion of $\omega^{a,0}(g,\lambda)$ ($\omega^{a,n}(g,\lambda)$ with $n \neq 0$) starts with the power λ^0 (λ^1). In fact, for $\omega^{a,0}(g)$ we have

$$\omega^{a,0}(g) = \left[\delta^{(a,0)}_{(b,n)} + \frac{1}{2!} f^{a,0}_{b,n\ c,m} g^{c,m} + o(g^2)\right] dg^{b,n}$$
$$= dg^{a,0} + \frac{1}{2!} f^{a,0}_{b,n\ c,m} g^{c,m} dg^{b,n} + o(g^3)$$
$$= dg^{a,0} + \frac{1}{2!}\left(f^{a,0}_{b,0\ c,0} g^{c,0} dg^{b,0} + f^{a,0}_{b,0\ c,n} g^{c,n} dg^{b,0}\right)$$
$$+ \frac{1}{2!}\left(f^{a,0}_{b,n\ c,0} g^{c,0} dg^{b,n} + f^{a,0}_{b,n\ c,m} g^{c,m} dg^{b,n}\right) + o(g^3) \qquad 35$$

which implies that under the rescaling $g^{a,0} \to g^{a,0},\ g^{a,n} \to \lambda g^{a,n}\ (n \neq 0)$,

$$\omega^{a,0}(g,\lambda) = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{a,0;\alpha}(g) \qquad 36$$

while for $\omega^{a,l}(g)$, with $l \neq 0$, we have

$$\omega^{a,l}(g) = \left[\delta^{(a,l)}_{(b,n)} + \frac{1}{2!} f^{a,l}_{b,n\ c,m} g^{c,m} + o(g^2)\right] dg^{b,n}$$

$$= dg^{a,l} + \frac{1}{2!} f^{a,l}_{b,n\ k,m} g^{c,m} dg^{b,n} + o(g^3)$$

$$dg^{a,l} + \frac{1}{2!}(f^{a,l}_{b,0\ c,n} g^{c,n} dg^{b,0} + f^{a,l}_{b,n\ c,0} g^{c,0} dg^{b,n} + f^{a,l}_{b,n\ c,m} g^{c,m} dg^{b,n}) + o(g^3). \quad (37)$$

Therefore the expansion of $\omega^{a,l}(g,\lambda)$ starts with the power λ^1

$$\omega^{a,n}(g,\lambda) = \sum_{\alpha=1}^{\infty} \lambda^\alpha \omega^{a,n;\alpha}(g). \quad (38)$$

However, for computation purposes it is better to spread the sum from zero and assume that $\omega^{a,n;0} = 0$ for $n \neq 0$. Thus we have that Eqs. (36),(38) can be summarized as:

$$\omega^{a,n}(g,\lambda) = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{a,n;\alpha}(g)$$

$$\omega^{a,n;0} = 0 \text{ for } n \neq 0. \quad (39)$$

Inserting (39) into the Maurer-Cartan equations $d\omega^{c,l} = -\frac{1}{2} f^{c,l}_{a,n\ b,m} \omega^{a,n} \omega^{b,m}$, we have

$$d\omega^{c,l;\alpha} = -\frac{1}{2} f^{(c,l;\alpha)}_{(a,n;\beta)\ (b,m;\gamma)} \omega^{a,n;\beta} \omega^{b,m;\gamma}$$

$$f^{(c,l;\alpha)}_{(a,n;\beta)\ (b,m;\gamma)} = \delta^\alpha_{\beta+\gamma} f^{c,l}_{a,n\ b,m} = \delta^\alpha_{\beta+\gamma} \delta^l_{n+m} f^c_{ab}$$

$$\omega^{a,n;0} = 0 \ \ for \ \ n \neq 0. \quad (40)$$

The case $\hat{\mathcal{G}} = V_0 \oplus V_1$ in which V_1 is a symmetric coset

It is possible to consider the infinite-dimensional algebra as $\hat{\mathcal{G}}=V_0 \oplus V_1$ where V_0 is generated by the infinite set of generators given by

$$\{\ldots, T_{a,-4}, T_{a,-2}, T_{a,0}, T_{a,2}, T_{a,4} \ldots\} \quad (41)$$

and where V_1 is generated by

$$\{\ldots, T_{a,-3}, T_{a,-1}, T_{a,1}, T_{a,3} \ldots\}. \quad (42)$$

From the commutation relation

$$[T_{a,n}, T_{b,m}] = f^{c}_{ab} T_{c,n+m} \quad (43)$$

we clearly see that the condition for a symmetric coset is to satisfy:

$$[V_0, V_0] \subset V_0$$
$$[V_0, V_1] \subset V_1$$
$$[V_1, V_1] \subset V_0. \quad (44)$$

It is therefore interesting to study the expansion of the infinite-dimensional algebra expanded with this choice of V_0 and V_1. For convenience we distinguish the generators $T_{a,n}$ where the index n is even from the case when the index is odd. The most natural choice is to use a subscript zero (one), $n_0 (n_1)$, for even values (odd). Thus (41–42) take the form:

$$\{T_{a,n_0}\} = \{\ldots, T_{a,-4}, T_{a,-2}, T_{a,0}, T_{a,2}, T_{a,4} \ldots\}, \tag{45}$$

$$\{T_{a,n_1}\} = \{\ldots, T_{a,-3}, T_{a,-1}, T_{a,1}, T_{a,3} \ldots\}, \tag{46}$$

$$\begin{aligned} [T_{a,n_0}, T_{b,m_0}] &= f_{ab}^{c} T_{c,n_0+m_0} = f_{a,n_0\ b,m_0}^{c,l_0} T_{c,l_0} \\ [T_{a,n_0}, T_{b,m_1}] &= f_{ab}^{c} T_{c,n_0+m_1} = f_{a,n_0\ b,m_1}^{c,l_1} T_{c,l_1} \\ [T_{a,n_1}, T_{b,m_1}] &= f_{ab}^{c} T_{c,n_1+m_1} = f_{a,n_1\ b,m_1}^{c,l_0} T_{c,l_0}. \end{aligned} \tag{47}$$

From where we see that the conditions of symmetric cosets for the structure constants are given by

$$f_{a,n_0\ b,m_0}^{c,l_1} = f_{a,n_0\ b,m_1}^{c,l_0} = f_{a,n_1\ b,m_1}^{c,l_1} = 0. \tag{48}$$

The idea is: (a) to find the expansions of $\omega^{i,n_0}(g,\lambda)$ and $\omega^{i,n_1}(g,\lambda)$; (b) to replace the expansions in the Maurer-Cartan equations and (c) to find the conditions under which are generated new algebras.

To find the expansions of $\omega^{a,n_0}(g,\lambda)$ and $\omega^{a,n_1}(g,\lambda)$ we must study the general expansion of $\omega^{a,n_0}(g)$ and $\omega^{a,n_1}(g)$ in terms of the coordinates and then analyze the behavior under the following rescaling:

$$\begin{array}{|c|}\hline g^{a,n_0} \to g^{a,n_0},\ g^{a,n_1} \to \lambda g^{a,n_1} \\ \hline n_0 = \ldots,-4,-2,0,2,4,\ldots \\ \hline n_1 = \ldots,-3,-1,1,3,\ldots. \\ \hline \end{array} \quad (49)$$

For $\omega^{a,n_0}(g)$ we find

$$\begin{aligned} \omega^{a,n_0}(g) &= \left[\delta^{(a,n_0)}_{(b,m)} + \frac{1}{2!} f^{a,n_0}_{b,m\ \ c,l} g^{c,l} + o(g^2)\right] dg^{b,m} \\ &= \delta^{(a,n_0)}_{(b,m)} dg^{b,m} + \frac{1}{2!} f^{a,n_0}_{b,m\ \ c,l} g^{c,l} dg^{b,m} + o(g^3) \\ &= dg^{b,n_0} + \frac{1}{2!} f^{a,n_0}_{b,m\ \ c,l} g^{c,l} dg^{b,m} + o(g^3) \\ &= dg^{b,n_0} + \frac{1}{2!} f^{a,n_0}_{b,m_0\ \ c,l_0} g^{c,l_0} dg^{b,m_0} + \frac{1}{2!} f^{a,n_0}_{b,m_1\ \ c,l_1} g^{c,l_1} dg^{b,m_1} + o(g^3). \end{aligned} \quad (50)$$

Analyzing higher order terms we find that if you rescale the parameters as in (49), then $\omega^{a,n_0}(g,\lambda)$ contains only even powers of λ. For this it is useful to write the condition as

$$f^{c,l_s}_{a,n_p\ \ b,m_q} = 0, \text{ for } s \neq (p+q) \bmod 2. \quad (51)$$

Performing the same procedure for $\omega^{a,n_1}(g,\lambda)$ we find that appear in the expansion only odd powers of λ. Thus we have

$$\omega^{a,n_0}(g,\lambda) = \sum_{\alpha=0}^{\infty} \lambda^{2\alpha} \omega^{a,n_0;2\alpha}(g)$$

$$\omega^{a,n_1}(g,\lambda) = \sum_{\alpha=0}^{\infty} \lambda^{2\alpha+1} \omega^{a,n_1;2\alpha+1}(g) \quad (52)$$

which can be written as

$$\omega^{a,n_p}(g,\lambda) = \omega^{a,n_{\bar{\alpha}}}(g,\lambda) = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{a,n_{\bar{\alpha}};\alpha}(g); \quad \bar{\alpha} = \alpha \bmod 2,\ p = 0,1. \quad (53)$$

Replacing (53) in the Maurer-Cartan equations, we obtain the following set of equations:

$$d\omega^{c,l_{\bar{\alpha}};\alpha} = -\frac{1}{2} f^{(c,l_{\bar{\alpha}};\alpha)}_{(a,n_{\bar{\beta}};\beta)(b,m_{\bar{\gamma}};\gamma)} \omega^{a,n_{\bar{\beta}};\beta} \omega^{b,m_{\bar{\gamma}};\gamma} \quad (54)$$

where

$$f^{(c,l_{\bar{\alpha}};\alpha)}_{(a,n_{\bar{\beta}};\beta)(b,m_{\bar{\gamma}};\gamma)} = f^{c,l_{\bar{\alpha}}}_{a,n_{\bar{\beta}}\ b,m_{\bar{\gamma}}} \delta^{\alpha}_{\beta+\gamma}$$

$$\bar{\alpha} = \alpha \bmod 2,\ \bar{\beta} = \beta \bmod 2,\ \bar{\gamma} = \gamma \bmod 2. \quad (55)$$

We find that the expanded algebra closes when the coefficients of the expansion are truncated at orders that satisfy the conditions

$N_1 = N_0 - 1$, or
$N_1 = N_0 + 1$.

56

Comment

We have shown that the expansion methods of the Maurer-Cartan 1-forms can be generalized so that they permit to study the expansion of the algebras of loops both when the compact finite-dimensional algebra $\mathcal{G}$ and the loop algebra (which is an infinite-dimensional algebra $\hat{\mathcal{G}}$ have a decomposition into two subspaces $V_0 \oplus V_1$.

- R. Caroca, N. Merino, P. Salgado, O. Valdivia, "Generating infinite dimensional algebras from loop algebras by expanding Maurer-Cartan forms, J. Math. Phys. 52, (2011)043519.