Exact Meron BH in four dimensional SU(2)Einstein Y-M theory

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14.03.2013

Self gravitating YM fields

- Usually in QCD gravitational interactions are ignored as they are too weak.
- There are situations where gravity effects are relevant for YM fields like in BH and near big bang.
- Simplest situation: e.o.m. derived from YM + EH action find exact solution

$$S[g_{\mu\nu}, A_\mu] = \int d^4x \sqrt{-g} \left(\frac{R - 2\Lambda}{\kappa} + \frac{1}{2e^2} Tr\left(F_{\mu\nu} F^{\mu\nu}\right) \right)$$

- Interesting for BH physics is that with YM field the “no hair theorem” does not apply. Only numerical solutions are known.
- Ansatz of spherically symmetric field strength which intuitivelly should be the easiest one is prohibitively difficult.

Embedded abelian solutions

- For an abelian gauge theory there exist a BH solution with electric and also magnetic charge (RN black hole)

$$
\begin{aligned}
ds^2 &= -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \\
& N = \sqrt{1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}}, \\
A &= -\frac{Q}{r} dt + P(1 - \cos\theta) d\phi,
\end{aligned}
$$

- The previous solution is also solution for the YM theory if we take

$$\mathbf{F}_{\mu\nu} = i\Pi_{\mu\nu}\sigma_3$$

- This means that we switch on only one YM generator
- This solution is trivial as with only one generator all the commutators in F=dA+[A,A] vanish and so all the features that distinguish YM from U(1) theory disappear
- These trivial solutions are called “embedded Abelian” and are up to now the only known exact solutions to the Einstein YM system

Non-trivial solutions

- All the ansatze to find non-trivial solutions (not only one generator switched on) failed for the ansatz of spherically symmetric field strength. As this is the simplest ansatz one may think that anything else will fail
- IDEA: In the Einstein equations enters only the stress tensor and not the field strength so that one can use a non spherically symmetric field strength leading to a sph. Symm. Stress tensor.
- One can choose a field strength which leads to a stress tensor of an abelian theory then the RN metric will be trivially solution
- However one must also solve the YM equations which remains non-trivial

Meron Ansatz

- In YM theory a gauge potential proportional to a pure gauge is NOT a pure gauge

$$\begin{aligned} A_\mu &= \lambda U^{-1}\partial_\mu U, \qquad \lambda \neq 0,1\,, \\ U &= U(x^\mu) \in SU(2)\,. \end{aligned}$$

$$F_{\mu\nu} = \lambda(\lambda - 1)\left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U\right]\,.$$

This a feature which is exclusive to non-abelian theories

The Ansatz simplifies the YM equations as a derivative drops out

$$
\begin{aligned}
U(x^\mu) &= Y^0\mathbf{1} + i\,Y^i\sigma_i, \qquad U^{-1}(x^\mu) = Y^0\mathbf{1} - iY^i\sigma_i\,, \\
Y^0 &= Y^0(x^\mu), \qquad Y^i = Y^i(x^\mu)\,, \\
\left(Y^0\right)^2 + Y^iY_i &= 1\,,
\end{aligned}
$$

$$
\begin{aligned}
A_\mu &= i\,\lambda P^k_\mu \sigma_k\,, \\
P^k_\mu &= \varepsilon_{ijk} Y_i \partial_\mu Y_j + Y^0 \partial_\mu Y^k - Y^k \partial_\mu Y^0
\end{aligned}
$$

Hedgehog Ansatz

- We make a “hedgehog” ansatz (at every point the field is outward pointing). This ansatz is spherically symmetric up to an internal rotation

$$\begin{aligned} U &= 1\cos f(r) + i\,\widehat{x}^i\sigma_i \sin f(r), \quad U^{-1} = 1\cos f(r) - i\,\widehat{x}^i\sigma_i \sin f(r), \\ \delta_{ij}\widehat{x}^i\widehat{x}^j &= 1, \end{aligned}$$

- In this ansatz the group element parametrization reads

$$\begin{aligned} Y^0 &= \cos f(r), \quad Y^i = \widehat{x}^i \sin f(r), \\ \widehat{x}^1 &= \sin\theta\cos\phi, \quad \widehat{x}^2 = \sin\theta\sin\phi, \quad \widehat{x}^3 = \cos\theta \end{aligned}$$

$$\begin{aligned} A_\mu &= i\lambda P_\mu^k \sigma_k, \\ P_\mu^k &= \sin^2 f \varepsilon_{ijk}\widehat{x}^i\partial_\mu\widehat{x}^j + \widehat{x}^k\partial_\mu f + \frac{\sin(2f)}{2}\partial_\mu\widehat{x}^k \end{aligned}$$

Solution of YM equation

- The previous anstaz is a solution when

$$\lambda = \frac{1}{2} \, .$$

$$f(r) = \frac{\pi}{2} \;\Rightarrow P^k_\mu = \varepsilon_{ijk} \widehat{x}^i \partial_\mu \widehat{x}^j \, .$$

$$\begin{aligned} F^i_{\mu\nu} &= Y^i \Pi_{\mu\nu} \, , \\ \Pi_{\mu\nu} &= 2\lambda(\lambda - 1) \left(\varepsilon_{mnq} Y^m \partial_{[\mu} Y^q \partial_{\nu]} Y^n \right) \end{aligned}$$

- The field strength factorizes in a radial Pauli matrix and and an abelian RN field strength

$$\frac{\Pi}{2\lambda(\lambda-1)} = -\frac{4\pi}{g} F_D = -\sin\theta d\theta \wedge d\phi \, ,$$

The metric

- As the field stregth factorizes in a hedgehog and an abelian effective part the RN metric is trivially a solution of the Einstein equations.
- However in opposition to the trivial case the magnetic charge is fixed

$$ds^2 = -\left(1 - \frac{\kappa M}{8\pi r} + \frac{4\kappa\lambda^2(\lambda-1)^2}{e^2 r^2} - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{dr^2}{1 - \frac{\kappa M}{8\pi r} + \frac{4\kappa\lambda^2(\lambda-1)^2}{e^2 r^2} - \frac{\Lambda r^2}{3}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

The non-abelianity of the solution

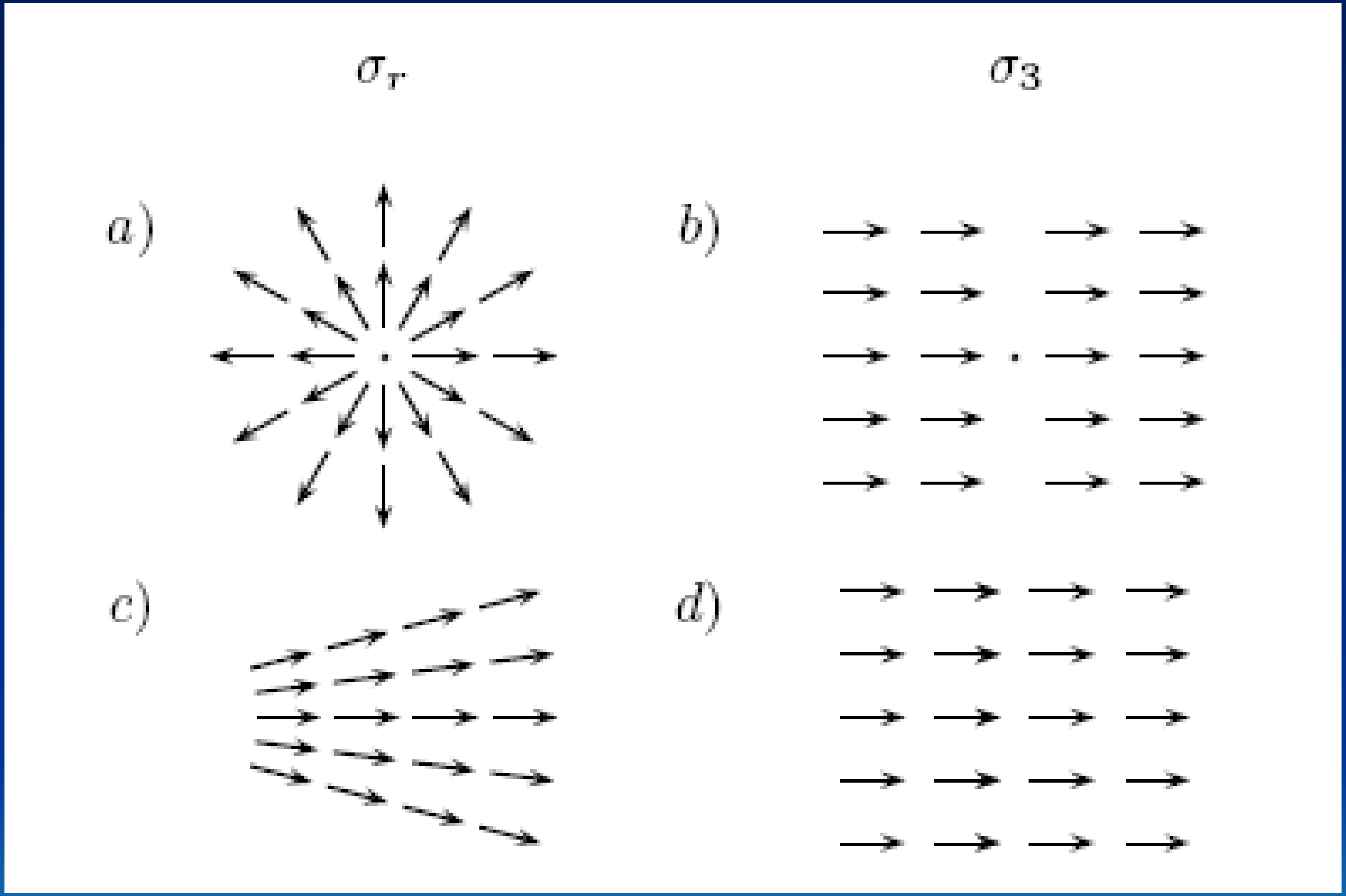
- Our field strength

$$
\begin{aligned}
\mathbf{F}_{\mu\nu} &= iF^i_{\mu\nu}\sigma_i = i\Pi_{\mu\nu}Y^i\sigma_i \equiv i\Pi_{\mu\nu}\sigma_r \\
i\Pi_{\mu\nu}Y^i\sigma_i &\equiv i\Pi_{\mu\nu}\sigma_r \ ,
\end{aligned}
$$

- Would be gauge equivalent to an abelian one if there was an EVERYWHERE well defined gauge transformation that brings it in

$$\mathbf{F}_{\mu\nu} = i\Pi_{\mu\nu}\sigma_3$$

- Such a gauge transformation must necessarily be singular at least on a point of a sphere of constant radius. Moreover the gauge transformation does not approach identity at infinity.
- The solution is only locally equivalent to an abelian but not globally. The two solutions belong to two different sectors of the solution space.


σ_r
σ_3
a)
b)
c)
d)

The non spherical symmetry of the field strength and its physical consequences

- The field strength is NOT spherically symmetric

$$Y^i \to R^i_j Y^j \qquad \sigma_r \to R^i_j Y^j \sigma_i \neq \sigma_r$$

- Due to the non-trivial mixing of space-time degrees of freedom with inner degrees of freedom of a hedgehog a space rotation must be compensated by an inner gauge rotation. This means that L is not a symmetry operator but we must take $\overrightarrow{J} = \overrightarrow{l} + \overrightarrow{\sigma}$

- We can study the e.o.m. of a SU(2) charged scalar field

$$g^{\mu\nu}\left(\nabla_\mu - A_\mu\right)\left(\nabla_\nu - A_\nu\right)\Phi = 0$$

$$\Phi = \exp\left(iEt\right)\psi\left(r,\theta,\phi\right)$$

$$\begin{aligned}\left(\nabla^\mu\nabla_\mu - a(r)\left(\vec{\sigma}\cdot\vec{l}\right) - b(r) + E^2\right)\psi &= 0\ ,\\ \vec{l} &= \vec{r}\times\vec{\nabla}\end{aligned}$$

- Half integer spin excitations also if all the fields involved are bosonic!!!!!

Non-Abelian charges

The classic definition of non-Abelian charge is in [27] (see also [28] [29]). The first step is to find a $SU(2)$-valued covariantly constant scalar ξ^i,

$$D_\mu \xi^i = 0 \ ,$$

where D_μ is the $SU(2)$ covariant derivative. Then, with this covariantly constant scalar one can construct fluxes which are conserved in the ordinary sense by contracting the field strength (or its dual) with ξ^i. In the present case, it is easy to see that the Y^i in Eq. (3.8) are covariantly constant with respect to the gauge field in Eqs. (3.3) and (3.5). Thus, following [27] and [29], the charge $Q = Q(Y^i)$ is the integral over the 2-sphere at infinity of the non-Abelian magnetic field B^i_μ contracted with Y^i,

$$Q = \frac{1}{4\pi}\int_{S^2_\infty} B^i_\mu Y_i n^\mu = -\frac{1}{8\pi}\int_{S^2_\infty} \Pi_{\theta\phi} d\theta d\phi = -\frac{1}{2} \ ,$$

It is interesting to note that if one would compute the non-abelian magnetic charges Q^i_M as surface integrals at infinity without projecting the magnetic field along the covariantly constant scalar Y_i as

$$Q_M = Q^i_M \sigma_i = \frac{1}{4\pi} \int_{S^2_\infty} F \ , \tag{5.8}$$

one would get a different result. In the case of the field strength in Eqs. (3.6), (3.7) and (3.8), the above expression reduces to

$$Q^i_M \sigma_i = \frac{1}{2\pi} \sigma_i \int_{S^2_\infty} Y^i(\theta,\phi) \Pi_{\theta\phi} d\theta d\phi \ , \tag{5.9}$$

and due to the presence of the functions $Y^i(\theta,\phi)$ (whose expressions are in Eq. (3.8)) the Q^i_M would vanish for all the components of the internal $su(2)$ index i.

However, here it is more appropriate the first approach. From the physical point of view, the idea to project the magnetic field along the Y_i corresponds to measure the charge with respect to the radial Pauli matrix defined in Eq. (5.3).