GENERALIZED HEDGEHOG and self-gravitating Skyrmions

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PLAN OF the Talk

- standard hedgehog in a covariant form
- NEW monopole BH with SKYRME effects
- Hedgehog without spherical symmetry
- Examples

Standard spherical hedgehog: SU(2);

$U = Y_0 \mathbb{1} + Y_i t^i$; $t^i t^j = -\delta^{ij} \mathbb{1} - \epsilon^{ijk} t^j t^k$;

$\mathbb{1} = 2 \times 2$ id. matrix ;

$U \in SU(2) \Leftrightarrow Y_0^2 + (Y_i)^2 = 1$

$$\boxed{S_D = k \int \sqrt{g}\, \mathrm{tr}\left[\left(U^{-1} \partial U\right)^2\right] d^D x}$$

Internal symmetries

$U \to (V^{-1} U V)$; where V is a constant element of $SU(2)$: $V \in SU(2)$

thus, this model has only global symmetries

S_D is very important in particles physics:

$u = \exp(i\pi_i(x^\mu)t^i)$, if the π_i are "small" then S_D describes Pions interactions (when $D=4$)

$$S_D \underset{|\pi| \ll 1}{\sim} K \int \frac{\sum_{i=1}^{3} (\partial_\mu \pi^i)^2}{2} + \underbrace{\text{interactions}}_{\sim O(\pi^3)}$$

With only one parameter one gets very good agreement with observations ...

higher order terms in the Taylor expansion in π^i

non-linear sigma model
also describes:

1) GRIBOV copies
2) Vortices in super-fluids
3) Some features of spin system
4) useful in diff. geometry in the theory of surfaces

AND MORE ...

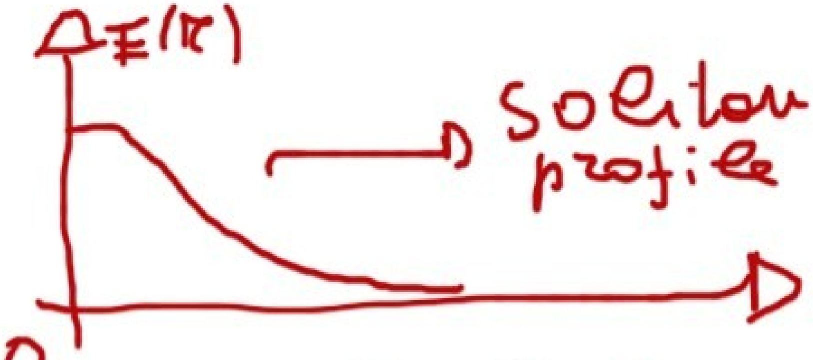
the SKYRME Lagrangian

$$S = S_D + \lambda \int \sqrt{g}\, d^4x \left(\mathrm{tr} \left\{ \left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U \right]^2 \right\} \right);$$

SKYRME (in the early sixties) introduced his famous term since S_D by itself does not support static finite energy configurations

$S_D: \quad \partial_\mu (U^{-1} \partial^\mu U) = 0 \; ;$

$\nexists \; U_{(0)}(x) \; | \; E^{(0)}(r)$ = energy density



soliton profile ? $\Rightarrow$ such that $E_{TOT} < +\infty$?

unfortunately ...

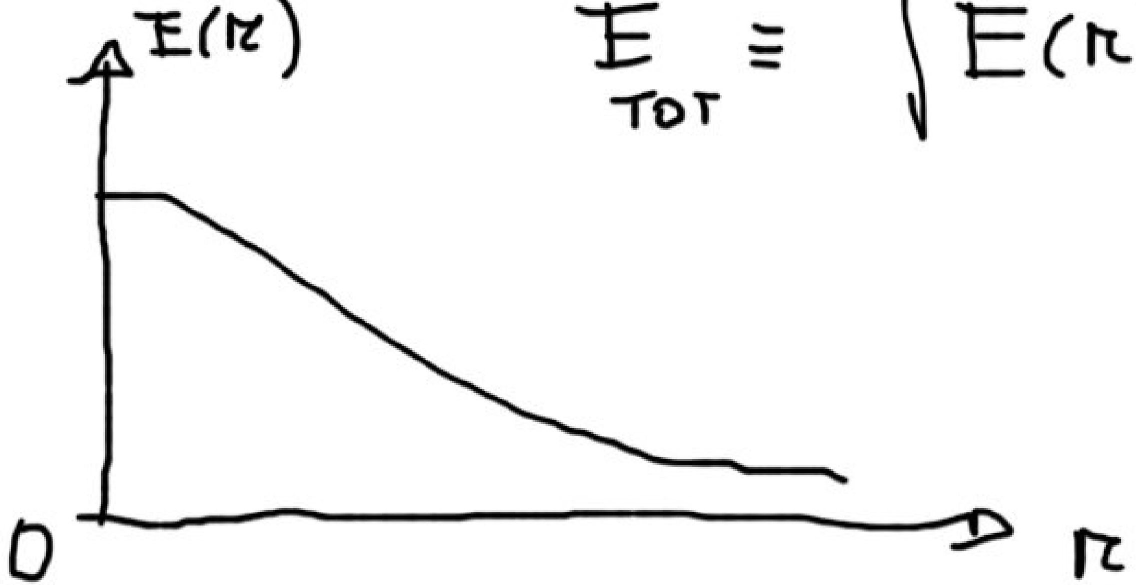
real situation $\Rightarrow$

divergent ...

With SKYRME...

$\exists\ U_{(0)}(x^\mu)$ such that

$E_{TOT} \equiv \int E(r) < \infty$



but $U_{(0)}(x^\mu)$ is not known analitically...

E. O. M. of SKYRME

$$\nabla^\mu R_\mu + \frac{\lambda}{4} \nabla^\mu \left[R^\nu, \left[R_\mu , R_\nu \right] \right] = 0$$

$R_\mu = U^{-1} \partial_\mu U$; λ is the SKYRME coupling

this is a matrix system of coupled non-linear P. D. E.

no exact solution is known, not even on flat spacetime...

A remarkable feature!!

Skyrme proposed to describe neutrons and protons as <u>FERMIONIC EXCITATIONS</u> around the SKYRME soliton.

One can have Fermionic excitations in a purely bosonic theory...

Pictorially ...

$\mathcal{U} \in SU(2)$; $\mathcal{U} = \exp(i\, t_i\, \pi^i(x^\mu))$

if $|\pi^i| \ll 1$ then $\pi^i \sim$ <u>Pions</u>

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if $\pi^i \sim \hat{\pi}^i + \psi^i$ where $\hat{\pi}^i$ is the SKYRME soliton

$\mathcal{U}_{(0)} = \exp(i\, t_i\, \hat{\pi}^i)$

and $|\psi^i| \ll 1$

then ψ^i behave as <u>Fermions</u>!

SKYRME Lagrangian is, for these reasons, one of the most important model in theoretical physics since it describes at the same time PIONS and nucleons.

ROUGH explanation:

Let $U_{(0)}$ be the SKYRME soliton the perturbations Ψ^i around $U_{(0)}$ satisfy an equation of this form:

$$0 = \Box \Psi^i + \lambda b(r) \vec{\sigma} \cdot \vec{l} \, \Psi^i + V(r) \Psi^i$$

these are the Pauli Matrices of SU(2) of SKYRME

This is the orbital angular momentum

$$\Rightarrow \boxed{\vec{J} = \vec{l} + \vec{\sigma}}$$

(ISO) Spin-orbit Coupling...

the term $\vec{\sigma}\cdot\vec{\ell}\,\Psi^i$ comes from the perturbations of the SKYRME term $\lambda\nabla^\mu\left[R^\nu,\left[R_\mu,R_\nu\right]\right]$ the mixing between internal and space-time symmetries generates the spin-flip !!

$\vec{\sigma} \cdot \vec{\ell}$ is the same as the "SPIN-ORBIT" coupling in Q.M.

$\vec{\sigma}$ are the generators of the internal symmetry group ($SU(2)$)

The spin-orbit coupling generates a spin-flip $\rightarrow$ the eigenvalues $\ell \rightarrow \ell \pm 1/2$

hence Ψ^i have half-integer spin.

thus, a bosonic theory can have Fermionic excitations!

In BH physics,
the SKYRME Lagrangian
allowed to construct the
first (numerical)
counterexamples to the NO-HAIR
conjecture.

OUR GOALS

We want to introduce nice ansatz which can provide one with more analitical results and also improve the numerical analysis. NICE ansatz are very welcome in a non-linear theory such as the SKYRME one...

useful identity:

$$\epsilon^{ijk}\epsilon^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m ;$$

standard definition:

$$R_\mu = R_\mu^i t_i = U^{-1}\nabla_\mu U ; \qquad t_i = -i\sigma_i ;$$

$$R_\mu^k = \epsilon^{ijk} y^i \nabla_\mu y^j + y^0 \nabla_\mu y^k - y^k \nabla_\mu y^0$$

$$U = \mathbb{1} y^0 + y^i t_i ; \quad (y^0)^2 + \sum_i^3 (y^i)^2 = 1 ;$$

SPHERICAL HEDGEHOG

$$Y_0 = \cos\alpha(r) \; ; \; Y_i = n_i \operatorname{sen}\alpha(r)$$

$$n_1 = \operatorname{sen}\theta \cos\varphi \; ; \; n_2 = \operatorname{sen}\theta \operatorname{sen}\varphi \; ;$$

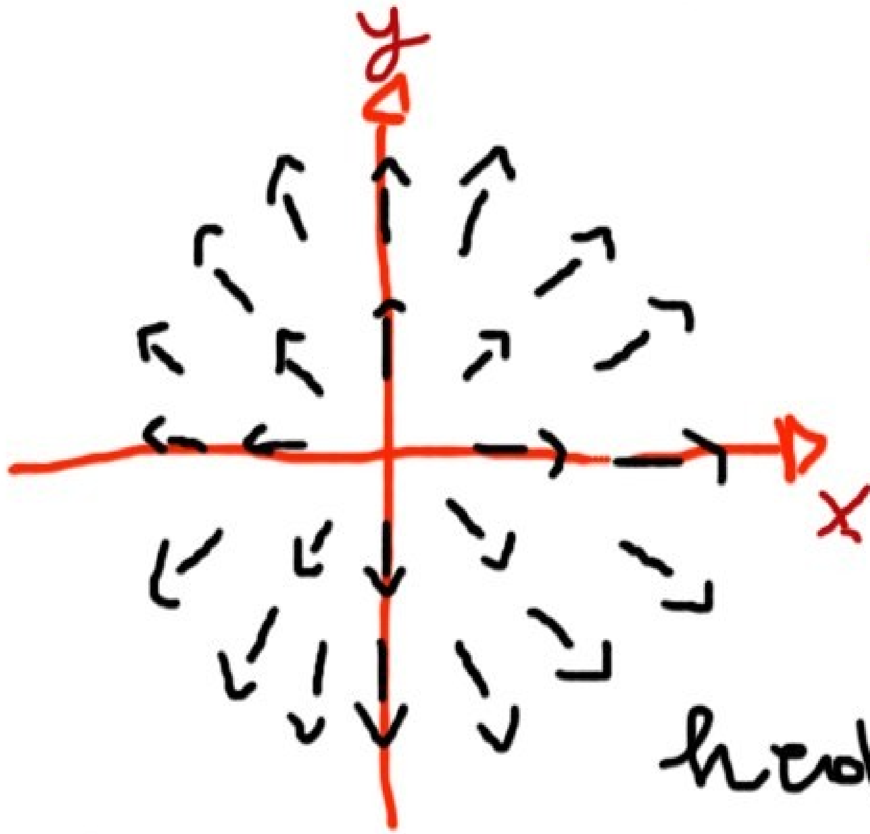
$$n_3 = \cos\theta \; ;$$

$$\Longrightarrow \quad U = \exp\left(i\alpha(r) n_i t^i\right)$$

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \operatorname{sen}^2\theta \, d\varphi^2\right)$$

the usual picture:

Physical space $\to$ (one dim. suppres.)



the arrows represent the direction in the internal space of the hedgehog config. in each spacetime point

$$U = \exp(i\alpha(r)\, n_i t^i) = \exp(i\alpha\, t_r);$$

$$\boxed{t_r = t^i n_i};$$

t_r is a "radial Pauli Matrix"

t_r is non-constant;

the arrows represent the internal direction of t_r in physical space.

Important property:

$U = \exp(i\alpha(r) t_r)$ may have winding number N:

$$N = \# \int d^3x \, \epsilon^{ijk} (U^{-1}\partial_i U)(U^{-1}\partial_j U)(U^{-1}\partial_k U)$$

this is a topological invariant which is related with the Baryon number of the SKYRME theory: but there is another property which is never stressed enough...

E.O.M. (non-linear sigma model)

$$\partial_\mu (U^{-1} \partial^\mu U) = 0$$ → 3 indep. P.D.E.

$$U^{-1} \partial_\mu U = R_\mu^i t_i$$

this is a nonlinear system of coupled P.D.E.s since U is matrix valued...

could be over-determined for $\alpha(r)$...

dangerous situation !!

$U = \exp(i\alpha(r)\, n_i t^i)$

and I plug into

$\partial_\mu(U^{-1}\partial^\mu U) = 0$ I would

have many equations for

only 1 unknown function

$\alpha(r)$

Covariant formulation
of the spherical hedgehog
and a new BH...

Let us first rewrite the
standard hedgehog in
a generic spherically symmetric
background in a intrinsic way

$$ds^2 = \underbrace{g_{AB}(y)\,dy^A dy^B}_{\text{METRIC in the ``r-t'' sector: we will call this 2-manifold } M_2.} + \rho^2(y)\,\underbrace{\gamma_{ab}\,dz^a dz^b}_{\text{Metric on the 2-Sphere } S^2}$$

ρ represents the scale of the S^2 factor

$A, B = 1, 2;$

$\alpha = \alpha(y);$

useful identity:

$$\epsilon^{ijk}\epsilon^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m \ ;$$

standard definition:

$$R_\mu = R_\mu^i t_i = U^{-1}\nabla_\mu U \ ; \qquad t_i = -i\sigma_i \ ;$$

$$R_\mu^k = \epsilon^{ijk} y^i \nabla_\mu y^j + y^0 \nabla_\mu y^k - y^k \nabla_\mu y^0$$

$$U = \mathbb{1} y^0 + y^i t_i \ ; \quad (y^0)^2 + \sum_i^3 (y^i)^2 = 1 \ ;$$

the SKYRME CASE

With the same notations:

$$0 = \left(1 + 2\lambda\rho^{-2}\,\mathrm{sen}^2\alpha\right) D^2\alpha + 2\rho^{-1}(D_A\rho)(D^A\alpha)$$

$$- \frac{\mathrm{sen}(2\alpha)}{\rho^2}\left[1 - \lambda\left((D\alpha)^2 - \frac{\sin^2\alpha}{\rho^2}\right)\right] ;$$

When $\lambda = 0$ it reduces to the previous E.O.M.

Difficult to solve in general but ...

a coupled system of PDEs for the profile α (which in principle overdetermines α) has reduced to a single scalar PDE...

Another Miracle of Holography!

$\alpha = \pi/2$ is always a solution both on flat and on curved spaces

the energy momentum tensor reads in this case

$$T_{\mu\nu} dx^\mu dx^\nu = -\frac{\kappa}{\rho^2}\left(1 + \frac{\kappa\lambda}{2\rho^{-2}}\right) g_{AB}\, dy^A dy^B +$$

$$+ \frac{\kappa\lambda}{2\rho^{-2}} \gamma_{ab}\, dz^a dz^b ;$$

NEW BH Solution

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right);$$

$$f(r) = \underbrace{1 - 8\pi G K}_{\text{deficit angle } \delta \text{ related to the non-linear sigma model;}} - \frac{2GM}{r} + \underbrace{\frac{4\pi G K \lambda}{r^2}}_{\text{SKYRME contribution;}} - \frac{\Lambda r^2}{3}$$

deficit angle δ
related to
the non-linear
sigma model;

SKYRME
contribution;

Some nice features...

1) the solution has just 1 integration constant M;

2) the coefficient of $1/r^2$ as well as the deficit angle are fixed in terms of the couplings...

Usual RN-Thermodynamics

$\delta M = \delta A + \Phi\, \delta Q \Longrightarrow$ in the RN case the BH is a solution for any choice of M and Q (so δM and δQ make sense)

While in our case we have to expect

$\boxed{\delta E = T\delta S}$ for some global mass E

$$T = \frac{1}{4\pi}\left(\frac{1-8\pi G K}{r_h} - \frac{4\pi G K\lambda}{r_h^3}\right)$$

$$S = \frac{\pi}{G} r_h^2 \; ; \; r_h = \frac{GM}{1-8\pi GM}\left(1+\sqrt{1-\frac{4\pi K\lambda(1-8\pi G K)}{GM^2}}\right)$$

$\hookrightarrow$ The ADM mass computed by Nucamendi-Sudarsky does not satisfy that relation: topological charge?

but we can integrate directly ...

$$\delta E = T\,\delta S \Rightarrow$$

integration constant

$$E = \frac{1}{2G}\left[(1-8\pi G K)\, r_h + \frac{4\pi G K \lambda}{r_h}\right] + E_0$$

$$\boxed{C \equiv \frac{dE}{dr_h} \cdot \left(\frac{dT}{dr_h}\right)^{-1}}$$

(unlike usual cases

interestingly thanks to SKYRME

$C > 0$ when $$\boxed{\frac{4\pi G K \lambda}{1-8\pi G K \lambda} < r_h^2 < \frac{12\pi G K \lambda}{1-8\pi G K \lambda}}$$

What a Miracle !!

a coupled system of non-linear PDEs reduced to a single scalar Master equation....
Which are the geometrical conditions which allow this miracle ???

GENERALIZED hedgehog

α is the profile of the hedgehog

these define the internal orientation

$$Y^0 = \cos\alpha \; ; \; Y^i = \operatorname{sen}\alpha \, n^i$$

$$\alpha = \alpha(x^\mu) \; ;$$

$$n^i = n^i(x^\mu) \; ; \quad \delta_{ij} n^i n^j = 1$$

$$R^i_\mu = \operatorname{sen}^2\alpha \, \epsilon^{ijk} n^j \partial_\mu n^k + n^i \partial_\mu \alpha + \frac{\operatorname{sen}(2\alpha)}{2} \partial_\mu n^i$$

Usual boundary conditions

$$\mathcal{U} = y_0 \mathbb{1} + y_i t^i = (\cos\alpha)\mathbb{1} + \sin\alpha\,(n_i t^i)$$

this defines the components along the generator of the internal symmetry

thus if $\alpha \xrightarrow[x^\mu \to \partial M]{} 0$ then

$$\mathcal{U} \xrightarrow[x^\mu \to \partial M]{} \mathbb{1}$$

if $\alpha \xrightarrow[x^\mu \to \partial M]{} \pi!$ then $\mathcal{U} \xrightarrow[x^\mu \to \partial M]{} -\mathbb{1}$

the differenre is that

α and Ω can now depend on any coordinates...

Which are the minimal geometric conditions allowing the existence of a hedgehog without spherical symmetry ??

COMPUTING E.O.M....

$$\partial^\mu R^i_\mu = \partial^\mu \left(\operatorname{sen}^2\alpha \, \epsilon^{ijk} n^j \partial_\mu n^k \right) +$$

$$+ n^i \Delta\alpha + (\partial_\mu n^i)(\partial_\mu \alpha) +$$

$$+ \frac{\operatorname{sen}(2\alpha)}{2} \Delta n^i +$$

$$+ (\partial_\mu n^i)\left(\partial^\mu \frac{\operatorname{sen}(2\alpha)}{2}\right);$$

In order for

$\boxed{\partial_\mu R^\mu = 0}$ to reduce to

a single scalar equation

it must happen that

$$(\partial_\mu R^\mu)^i = (P(\alpha))\, n^i$$

namely $\partial_\mu R^\mu = P(\alpha)\, n^i t_i$

for some scalar $P(\alpha)$ ...

Sufficient conditions.

1) $(\partial_\mu \alpha)(\partial_\mu n^i) = 0$;

2) $\Delta n^i = l\, n^i$

where l can be a function of the coordinates but it must be the same for all non-vanishing n^i

these conditions are satisfied for the standard spherical hedgehog and they allow to generalize it without spherical symmetry!

GENERALIZED HEDGEHOG!

$Y^0 = \cos\alpha \; ; \; Y^i = \mathrm{sen}\,\alpha \, n^i \; ;$

$n^1 = \cos\Theta \; ; \; n^2 = \mathrm{sen}\,\Theta \; ; \; n^3 = 0 \; ;$

$$\boxed{\nabla_\mu \alpha \nabla^\mu \Theta = 0 \; ; \quad \Delta \Theta = 0 \; ;}$$

$\underline{\alpha}$ is the profile of the hedgehog;

$\underline{n}^i$ (and thus $\underline{\Theta}$) define the orientation in the internal space;

Example 1 : $\boxed{T^3 \times \mathbb{R}}$

$$ds^2 = -d\tau^2 + d\varphi_1^2 + d\varphi_2^2 + d\varphi_3^2$$

$$\begin{cases} \alpha = \alpha(\varphi_1, \tau)\,; \\ n^1 = \cos(a\varphi_2 + b\varphi_3)\,; \\ n^2 = \operatorname{sen}(a\varphi_2 + b\varphi_3)\,; \quad n^3 = 0\,; \end{cases}$$

E.O.M. reduces to (γ constant)

$$\Box\alpha + \gamma \operatorname{sen}(2\alpha) = 0 \longrightarrow \underline{\text{integrable}}$$

the SKYRME case!

Similar procedure : E.O.M.

$$\nabla_\mu R^\mu + \lambda \underbrace{\nabla^\mu \left[R^\nu , \left[R_\mu , R_\nu \right] \right]}_{\Delta} = 0 \;;$$

We have to study under which conditions the term is proportional to $n^i\, S(\alpha)$ for some $S(\alpha)$...

the EXTRA conditions

3) $(\nabla^\mu \nabla^\nu \Theta)(\nabla_\mu \Theta)(\nabla_\nu \Theta) = 0$;

4) $(\nabla^\mu \nabla^\nu \alpha)(\nabla_\mu \alpha)(\nabla_\nu \Theta) = 0$;

we will now present many examples in which this condition are verified

(3) basically implies that Θ depends linearly on the Killing coordinates...

Example 1 with SKYRME

plane and hyperbolic symmetry

$$ds^2 = g_{AB}(y)\, dy^A dy^B + r^2(y)\, \underbrace{\gamma_{ab}\, dz^a dz^b}$$

$\to d\theta^2 + d\phi^2$

$\to d\theta^2 + \sinh^2\theta\, d\phi^2$

I) $\alpha = \alpha(\rho)$; $\Theta = \omega t$;

$$ds^2 = -f(\rho)\, dt^2 + g(\rho)\, d\rho^2 + h(\rho)\, \gamma_{ab}\, dz^a dz^b$$

II) $\alpha = \alpha(t)$; $\Theta = \omega\rho$

$$ds^2 = -a(t)\, dt^2 + b(t)\, d\rho^2 + c(t)\, \gamma_{ab}\, dz^a dz^b$$

Example 2: Weyl-Papapetrou

$$ds^2 = -A\exp\left(\frac{\Omega}{2}\right)(dt + \omega d\varphi)^2 + \frac{A\, d\varphi^2}{\exp\left(\frac{\Omega}{2}\right)}$$

$$+ \frac{\exp(2\nu)}{\sqrt{A}}\left(dr^2 + dz^2\right);$$

A, Ω, ω and ν depend on r and z;

$\alpha = \alpha(r, z)$; $\Theta = \gamma t + m\varphi$;

Still, even if the non-linear sigma model depends explicitely on Killing coordinates, $T_{\mu\nu}$ does not!!

NON-TRIVIAL REALIZATIONS OF SYMMETRIES!!

the generalized hedgehog allows to realize spacetime symmetries in a non-trivial way: $\exists\, \xi \mid \; L_{\xi} g_{\mu\nu} = 0$ and

$L_{\xi} U \neq 0$ but in such a way that $L_{\xi} T_{\mu\nu}(U) = 0$!

even if $L_{\xi} U = 0$ one can have $L_{\xi} T_{\mu\nu}(U) = 0$

Last example

$\alpha = \pi/2$; $T^z_z - T^r_r = 0 = T^t_t + T^\varphi_\varphi$;

$T^\varphi_t = K\rho A^{-1} \exp\left(\frac{\Omega}{2}\right)(m - p\omega)$;

$T^z_z + T^r_r = -K \dot{A}^{-1} \exp\left(\frac{\Omega}{2}\right)\left(-p^2 \exp(-\Omega) + (m - p\omega)^2\right)$;

$T^t_t - T^\varphi_\varphi = -K \dot{A}^{-1} \exp\left(\frac{\Omega}{2}\right)\left(\frac{p^2}{\exp(\Omega)} + m^2 - (p\omega)^2\right)$;

At a first glance, the same $T_{\mu\nu}$ can be realized by a KG field: $\boxed{\Psi = p_1 t + p_2 \varphi}$ but ...

In our case, by construction, the SKYRME field satisfies

$U(\varphi) = U(\varphi + 2\pi)$; a linear KG $\psi = P_1 t + P_2 \varphi$ does not!

even if one takes $P_2 = 0$

then $\psi = P_1 t$ is unbounded

and therefore unphysical

Conclusions

1) the generalized hedgehog is a very powerful ansatz

2) It allows to simplify the Skyrme E.O.M. to a simple scalar equation

3) Open new secters without spherical symmetry

4) non-trivial realizations of symmetries

5) new BH and other taub-NUT solutions...