# RÉNYI AND ENTANGLEMENT ENTROPIES: some holographically calculable contributions

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Glimpse at AdS/CFT Entanglement entropy Holographic derivation Rényi entropy Outlook

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- 1. anything  $Conformal_d \Leftrightarrow something in AdS_{d+1}$
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- ► (Dirac'36, "Wave equations in conformal space") conformal group in Minkowski<sub>d</sub> 

  isometries of (A)dS<sub>d+1</sub>
- ► (Fefferman-Graham'85, "Conformal invariants") conformal manifold M<sub>d</sub> ⇔ conformal infinity of Poincaré metric
- ► (Maldacena'97, 15 years of AdS/CFT correspondence) conformal field theory ⇔ string/M-theory, (quantum) gravity





#### Outline

Motivation

Glimpse at AdS/CFT

Entanglement entropy

Holographic derivation

Rényi entropy

Outlook





### Perfect sense: extracts from AdS/CFT dictionary

- 't Hooft coupling  $\Leftrightarrow$  string length:  $\lambda^{-1/4} = I_s$
- ▶ rank of the gauge group  $\Leftrightarrow$  Planck length :  $N^{-1/4} = I_P$
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- ▶ energy scale ⇔ radial direction (IR-UV connection)
- ▶ trace anomaly ⇔ volume anomaly (Henningson+Skenderis'98)
- ▶ entanglement entropy ⇔ area of minimal surface (Ryu+Takayanagi'06)





#### A saucerful of secrets: entanglement

Entanglement: measure of "quantum-ness" (EPR paradox and Schrödinger's cat, 1935...and today's quantum information theory)

Entanglement/geometric entropy: von Neumann entropy of reduced density matrix

$$\rho_{\rm A} = {\it tr}_{\rm B} \{ \rho_{{\it A} \oplus {\it B}} \} \qquad {\it S}_{\rm EE} = -{\it tr}_{\it A} \{ \rho_{\it A} \, \ln \rho_{\it A} \}$$





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Area law (Srednicki'93) : free fields in d spatial dimensions with spherical entangling surface

$$S_{\textit{EE}} \sim rac{\textit{Area of entangling surface}}{\epsilon^{\textit{d}-1}}$$

Of interest in black hole physics: very reminiscent of Bekenstein-Hawking area law





Motivation

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Universal scaling at one-dimensional (d=1) conformal critical points (Holzhey+Larsen+Wilczek'94): probably the most ubiquitous formula in last decade's literature

$$\mathcal{S}_{\scriptscriptstyle \it EE} = rac{m{c}}{3} \cdot \ln \, rac{\ell}{\epsilon}$$





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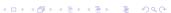
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- Regularized geodesic length:  $L \ln \frac{\ell^2}{\epsilon^2}$
- Need entry of AdS/CFT dictionary (Brown+Henneaux'86): central charge of the algebra of asymptotic diffs (PBH) in AdS<sub>3</sub>

$$c = \frac{3L}{2G_{\Lambda}}$$



#### Obscured by clouds: EE for free fields, higher dims,... ⇔ ???

Universality: for d > 1 the leading area term is reg-scheme-dependent

Conundrum: for even d + 1, a universal (and conformally invariant) logarithmic term with Type-A trace anomaly coefficient a in front.

Established numerically (Lohmayer+Neuberger+Schwimmer+Theisen'09) as well as analytically (Dowker'10).





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Despite many efforts (Myers,Sinha,Casini,Huerta,Fursaev,...), no clear holographic picture as yet.





#### Wearing the inside out: a holographic derivation in two steps

 First (e.g. free scalar): EE as thermal entropy in S<sup>1</sup> × H<sup>d</sup> (Casini+Huerta'10)

via conformal maps reduces to the computation of 1-loop effective action for a conformal scalar, i.e., determinant of the conformal Laplacian (Yamabe operator, Y)

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Second: invoke now the holographic formula (DD'08-09,Aros+DD'10):

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boundary  $M = S^1 \times H^d$ , bulk  $X = H^{d+2}$ 

evaluate for a special value of the mass of the bulk field

determinant and trace anomaly  $\Leftrightarrow$  1-loop effective action for bulk duals





holographic formula at work

Green is the colour:



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Evaluation of the quantum 1-loop effective actions in the bulk using the resolvent or Green function: look for  $\hbar$  effect in the bulk !!!





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$$S_q^{(R)} = \frac{1}{1-q} \ln tr \rho^q$$

$$\mathcal{S}_1^{(R)}=\mathcal{S}_{_{EE}}$$





Outlook

#### A saucerful of secrets: Rényi entropy / q-entropies

Rényi vs. Shannon:

$$S_q^{(R)} = \frac{1}{1-q} \ln tr \rho^q$$
  $S_1^{(R)} = S_{El}$ 

Mutatis mutandis

- boundary  $M = S^1 \times H^d$  with different radii, bulk  $X = H^{d+2}$  with conical singularity
- use Sommerfeld formula to work out the Green function/ heat kernel
- example, d = 1:

$$S_q^{(R)} = \frac{c}{6} \cdot (1 + \frac{1}{q}) \cdot \ln \frac{\ell}{\epsilon}$$





#### What shall we do now: Outlook

- spinor: mutatis mutandis √; higher spins: ???
- connection with entropy of extremal black holes (Solodukhin)
- geometry of entangling surface and type-B trace anomaly: Wald or not Wald (Myers et al.)
- guess: finite contributions (Klebanov et al.) ⇔ q-deformed Patterson-Selberg zeta (Floyd L. Williams)
- c-theorem, candidate for odd dimensions (Ryu+Takayanagi, Myers+Sinha, Casini+Huerta,...)



